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Abstract. Let $\Delta$ be the open unit disc in $\mathbb{C}$, $X$ a connected complex manifold and $D$ the set of all holomorphic maps $f: \Delta \to X$ with $\overline{f(\Delta)} = X$. We prove that $D$ is dense in $\text{Hol}(\Delta, X)$.

1. Introduction

Let $\Delta_r = \{ z \in \mathbb{C} : |z| < r \}$ and $\Delta = \Delta_1$. In [7] the second author proved that for any irreducible complex space $X$ there exists a holomorphic map $\Delta \to X$ with dense image, and he raised the question whether the set of all holomorphic maps $\Delta \to X$ with dense image forms a dense subset of the set $\text{Hol}(\Delta, X)$ of all holomorphic maps $\Delta \to X$ with respect to the topology of locally uniform convergence.

In this paper we show that the answer to this question is positive if $X$ is smooth, but negative for some singular space.

Theorem 1. For any connected complex manifold $X$ the set of holomorphic maps $\Delta \to X$ with dense images forms a dense subset in $\text{Hol}(\Delta, X)$. The conclusion fails for some singular complex surface $X$.

The situation is quite different for proper discs, i.e., proper holomorphic maps $\Delta \to X$. The paper [3] contains an example of a non-pseudoconvex bounded domain $X \subset \mathbb{C}^2$ such that a certain nonempty open subset $U \subset X$ is not intersected by the image of any proper holomorphic disc $\Delta \to X$. On the other hand, proper holomorphic discs exist in great abundance in Stein manifolds [5], [1], [2].

2. Preparations

Lemma 1. Let $W_n$ be a decreasing sequence (i.e., $W_{n+1} \subset W_n$) of open sets with $\Delta \subset W_n \subset \Delta_2$ for every $n$. Let $K = \cap_n \overline{W_n}$ and assume that the interior of $K$ coincides with $\Delta$. Furthermore assume that there are biholomorphic maps $\phi_n : \Delta \to W_n$ with $\phi_n(0) = 0$ for $n = 1, 2, \ldots$.
Then there exists an automorphism \( \alpha \in \text{Aut}(\Delta) \) and a subsequence \((\phi_{n_k})\) of the sequence \((\phi_n)\) such that \(\phi_{n_k} \circ \alpha^{-1}\) converges locally uniformly to the identity map \(\text{id}_\Delta\) on \(\Delta\).

Proof. Montel’s theorem shows that, after passing to a suitable subsequence, we have \(\lim_{n \to \infty} \phi_n = \alpha: \Delta \to K\) and \(\lim_{n \to \infty} (\phi_n^{-1}|_\Delta) = \beta: \Delta \to \Delta\). Since the limit maps are holomorphic and satisfy \(\alpha(0) = 0\) and \(\beta(0) = 0\), we conclude that \(\alpha(\Delta) \subset \text{Int}K = \Delta\) and \(\beta(\Delta) \subset \Delta\). Moreover \(\alpha \circ \beta = \text{id}_\Delta = \beta \circ \alpha\), and hence both \(\alpha\) and \(\beta\) are automorphisms of \(\Delta\) (indeed, rotations \(z \to z e^{i\theta}\)). \(\square\)

We also need the following special case of a result of the first author (theorem 3.2 in [4]):

**Proposition 1.** Let \(X\) be a complex manifold, \(0 < r < 1\), \(E\) the real line segment \([1, 2] \subset \mathbb{C}\), \(K = \overline{x} \cup E\), \(U\) an open neighbourhood of \(\overline{x}\) in \(\mathbb{C}\), \(S\) a finite subset of \(K\) and \(f: U \cup E \to X\) a continuous map which is holomorphic on \(U\).

Then there is a sequence of pair of open neighbourhoods \(W_n \subset \mathbb{C}\) of \(K\) and holomorphic maps \(g_n: W_n \to X\) such that:

1. \(g_n|_K\) converges uniformly to \(f|_K\) as \(n \to \infty\), and
2. \(g_n(a) = f(a)\) for all \(a \in S\) and \(n \in \mathbb{N}\).

3. **Towards the main result**

In this section we prove the following proposition which is the main technical result in the paper. The first statement in theorem 1 (§1) is an immediate corollary.

**Proposition 2.** Let \(X\) be a connected complex manifold endowed with a complete Riemannian metric and induced distance \(d\), \(S\) a countable subset of \(X\), \(f: \Delta \to X\) a holomorphic map, \(\epsilon > 0\) and \(0 < r < 1\).

Then there exists a holomorphic map \(F: \Delta \to X\) such that

(a) \(S \subset F(\Delta)\), and
(b) \(d(f(z), F(z)) \leq \epsilon\) for all \(z \in \Delta_r\).

Proof. Let \(s_1, s_2, s_3, \ldots\) be an enumeration of the elements of \(S\). We shall inductively construct a sequence of holomorphic maps \(f_n: \Delta \to X\), numbers \(r_n \in (0, 1)\) and points \(a_{1,n}, \ldots, a_{n,n} \in \Delta\) satisfying the following properties for \(n = 0, 1, 2, \ldots:\)

1. \(f_0 = f\) and \(r_0 = r\),
2. \((r_n + 1)/2 < r_{n+1} < 1\),
3. \(f_n(a_{j,n}) = s_j\) for \(n \geq 1\) and \(j = 1, 2, \ldots, n\),
4. \(d(f_n(z), f_{n+1}(z)) < 2^{-(n+1)}\epsilon\) for all \(z \in \Delta_{r_n}\), and
(5) $d_{\Delta}(a_{j,n}, a_{j,n+1}) < 2^{-n}$ for $j = 1, 2, \ldots, n$ where $d_{\Delta}$ denotes the Poincaré distance on the unit disc.

Assume inductively that the data for level $n$ (i.e., $f_n, r_n, a_{j,n}$) have been chosen. (For $n = 0$ we do not have any points $a_{j,0}$.) With $n$ fixed we choose an increasing sequence of real numbers $\lambda_k$ with $\lambda_k > r_n$ and $\lim_{k \to \infty} \lambda_k = 1$. For every $k \in \mathbb{N}$ the map $g_k(z) \overset{def}{=} f_n(\lambda_k z) \in X$ is defined and holomorphic on the disc $\Delta_k = \Delta \cap \mathbb{C}$. After a slight shrinking of its domain we can extend it continuously to the segment $E = [1, 2] \subset \mathbb{C}$ such that the right end point 2 of $E$ is mapped to the next point $s_{n+1} \in S$ (this is possible since $X$ is connected).

Applying proposition 1 to the extended map $g_k$ we obtain for every $k \in \mathbb{N}$ an open neighbourhood $V_k \subset \mathbb{C}$ of $K = \Delta \cup E$ and a holomorphic map $g_k : V_k \to X$ such that

(i) $|g_k(z) - f_n(\lambda_k z)| < 2^{-k}$ for all $z \in \Delta$,
(ii) $g_k(2) = s_{n+1}$, and
(iii) $g_k(a_{j,n}/\lambda_k) = f_n(a_{j,n}) = s_j$ for $j = 1, \ldots, n$.

Next we choose a decreasing sequence of simply connected open sets $W_k \subset \mathbb{C}$ ($k \in \mathbb{N}$) with $K \subset W_k \subset V_k$ and $K = \cap_k \overline{W_k}$. Notice that Int $K = \Delta$. By lemma 1 there is a sequence of biholomorphic maps $\phi_k : \Delta \to W_k$ with $\lim_{k \to \infty} \phi_k = i d_{\Delta}$.

Consider the holomorphic maps $h_k = g_k \circ \phi_k : \Delta \to X$. By our construction we know that $\lim_{k \to \infty} h_k = f_n$ locally uniformly on $\Delta$.

To fulfill the inductive step it thus suffices to choose $f_{n+1} = h_k$ for a sufficiently large $k$, $a_{j,n+1} = a_{j,n}/\lambda_k$ ($j = 1, \ldots, n$), $a_{n+1,n+1} = \phi_k^{-1}(2)$. Finally we choose a number $r_{n+1}$ satisfying

$$\max\{|a_{n+1,n+1}|, \frac{r_n + 1}{2}\} < r_{n+1} < 1.$$ 

This completes the inductive step.

By properties (2) and (4) the sequence $f_n$ converges locally uniformly in $\Delta$ to a holomorphic map $F : \Delta \to X$. Aided by property (1) we also control $d(f(z), F(z))$ for $z \in \Delta$. Since the Poincaré metric is complete, property (5) insures that for every fixed $j \in \mathbb{N}$ the sequence $a_{j,n} \in \Delta$ ($n = j, j + 1, \ldots$) has an accumulation point $b_j$ inside of $\Delta$, and (3) implies $F(b_j) = s_j$ for $j = 1, 2, \ldots$. Hence $S \subset F(\Delta)$. \hfill \Box

4. Singular spaces

We use an example of Kaliman and Zaidenberg [6] to show that for a complex spaces $X$ with singularities the set of maps $\Delta \to X$ with dense image need not be dense in $Hol(\Delta, X)$. We denote by $Sing(X)$ the singular locus of $X$. 
Proposition 3. There is a singular compact complex surface $S$, a non-constant holomorphic map $f: \Delta \to S$ and an open neighbourhood $\Omega$ of $f$ in $\text{Hol}(\Delta, S)$ such that $g(\Delta) \subset \text{Sing}(S)$ for every $g \in \Omega$.

Proof. In [6] Kaliman and Zaidenberg constructed an example of a singular surface $S$ with normalization $\pi: Z \to S$ such that $S$ contains a rational curve $C \cong \mathbb{P}^1$ while $S$ is smooth and hyperbolic. Denote by $d_Z$ the Kobayashi distance function on $Z$. We choose two distinct points $p, q \in C$ and open relatively compact neighbourhoods $V$ of $p$ and $W$ of $q$ in $S$ such that $\overline{V} \cap \overline{W} = \emptyset$. The preimages $\pi^{-1}(\overline{V})$ and $\pi^{-1}(\overline{W})$ in $Z$ are also compact, and since $Z$ is hyperbolic we have

$$r = \min \{d_Z(x, y) : x \in \pi^{-1}(\overline{V}), y \in \pi^{-1}(\overline{W})\} > 0.$$  

Fix a point $a \in \Delta$ with $0 < d_\Delta(0, a) < r$ and let $\Omega$ consist of all holomorphic maps $g: \Delta \to S$ satisfying $g(0) \in V$ and $g(a) \in W$. Since both $p$ and $q$ are lying on the rational curve $C$, there is a holomorphic map $g: \Delta \to C$ with $g(0) = p \in V$ and $g(a) = q \in W$; hence the set $\Omega$ is not empty. Clearly $\Omega$ is open in $\text{Hol}(\Delta, S)$.

To conclude the proof it remains to show that $g(\Delta) \subset \text{Sing}(S)$ for all $g \in \Omega$. Indeed, a holomorphic map $g: \Delta \to S$ with $g(\Delta) \not\subset \text{Sing}(S)$ admits a holomorphic lifting $\bar{g}: \Delta \to Z$ with $\pi \circ \bar{g} = g$. If $g \in \Omega$ then by construction

$$d_Z(\bar{g}(0), \bar{g}(a)) \geq r > d_\Delta(0, a)$$

which violates the distance decreasing property for the Kobayashi pseudometric. This contradiction establishes the claim.  

In particular, we see that in this example the set of all holomorphic maps $f: \Delta \to S$ with dense image does not constitute a dense subset of $\text{Hol}(\Omega, S)$.  

References


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