FOUR GRAVITY RESULTS

Z. Dvořák     R. Škrekovski     T. Valla

ISSN 1318-4865

July 19, 2004

Ljubljana, July 19, 2004
Four Gravity Results\textsuperscript{1}

Z. Dvořák\textsuperscript{2}, R. Škrekovski\textsuperscript{3}, T. Valla\textsuperscript{2}

June 29, 2004

\textsuperscript{2} Charles University, Faculty of Mathematics and Physics, DIMATIA and Institute for Theoretical Computer Science (ITI) Malostranské nám. 2/25, 118 00, Prague, Czech Republic 
{rakdver, valla}@kam.mff.cuni.cz

\textsuperscript{3} Department of Mathematics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia 
skreko@fmf.uni-lj.si

Abstract

The gravity of a graph $H$ in a given family of graphs $\mathcal{H}$ is the greatest integer $n$ with the property that for every integer $m$, there exists a supergraph $G \in \mathcal{H}$ of $H$ such that each subgraph of $G$, which is isomorphic to $H$, contains at least $n$ vertices of degree $\geq m$ in $G$. Madaras and Škrekovski introduced this concept and showed that the gravity of the path $P_k$ on $k \geq 2$ vertices is $k - 2$ for each $k \neq 5, 7, 8, 9$. They conjectured that for each of the four excluded cases the gravity is $k - 3$. In this paper we show that this holds.

1 Introduction

Throughout the paper, we consider connected graphs without loops or multiple edges. Let $\mathcal{H}$ be a family of graphs, and let $H$ be a connected graph

\textsuperscript{1}Supported in part by bilateral project SLO-CZ/04-05-002 between Slovenia and Czech Republic
\textsuperscript{2}Supported in part by bilateral project MSMT-07-0405 between Slovenia and Czech Republic
\textsuperscript{3}Supported in part by Research Project Z1-3129 of the Ministry of Science and Technology of Slovenia
such that infinitely many members of $\mathcal{H}$ contain a subgraph isomorphic to $H$. Let $\varphi(H, \mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains a subgraph isomorphic to $H$, contains also a subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$\deg_G(v) \leq \varphi(H, \mathcal{H}).$$

If such a finite $\varphi(H, \mathcal{H})$ does not exist, we write $\varphi(H, \mathcal{H}) = +\infty$. We say that the graph $H$ is light in the family $\mathcal{H}$ if $\varphi(H, \mathcal{H}) < +\infty$, otherwise we call it heavy. Thus, $H$ is heavy in $\mathcal{H}$ if, for every integer $m$, there is a graph $G \in \mathcal{H}$ such that each isomorphic copy of $H$ in $G$ contains a vertex of degree $m$ or in $G$.

It is well known that every plane graph contains a vertex of degree at most 5. Kotzig [9] stated that each 3-connected plane graph contains an edge of weight at most 13 and at most 11 in the case of absence of 3-vertices, and these bounds are sharp. This result was generalized in many directions; namely, it served as starting point for looking for subgraphs of small weight in plane graphs.

Borodin [2] extended Kotzig’s theorem by showing that every simple planar graph with minimum degree $\geq 3$ has also an edge of weight $\leq 13$. This extension of Kotzig’s theorem will be applied in few proofs in this paper. Fabrici and Jendrol’ [3] proved that the only light graphs in the family of all 3-connected plane graphs are paths; this holds also for the family of all 3-connected plane graphs of minimum degree 4 (see [4]) and of minimum face size 4 (see [6]). In the family of plane graphs of minimum degree 5, there are light graphs other than paths [1, 5, 7, 13]. The lightness of paths and some other graphs in various families of planar graphs was studied also in [10, 12]. The survey of results on light graphs in various families of plane, projective plane, and general graphs can be found in the paper Jendrol’ and Voss [8].

The gravity of a connected graph $H$ in the family $\mathcal{H}$ of planar graphs is the greatest integer $n$ with the property that for every integer $m$ there exists a supergraph $G \in \mathcal{H}$ such that each subgraph of $G$, which is isomorphic to $H$, contains at least $n$ vertices of degree $\geq m$ in $G$. Hence, a graph is light in a family of graphs if and only if its gravity is zero.

The concept of gravity was introduced by Mačanar and Škrekovski [11]. They determined the gravity of stars in the class $\mathcal{P}_d$ of planar graphs of minimum degree $\geq d$ for $d \in \{1, \ldots, 5\}$. In $\mathcal{P}_1$ the gravity of the path $P_k$ is $k - 1$ for each $k \neq 3, 5$ and it is $k - 2$ for $k = 3, 5$. In $\mathcal{P}_2$ the gravity of the
path $P_k$ is at most $k - 2$ for each $k \geq 2$ and the gravity reaches the bound of $k - 2$ for each $k \neq 5, 7, 8, 9$. Skrekovski and Madaras conjectured that for each of the four excluded cases the gravity is $k - 3$. In this paper we show that this holds. Our arguments are based on the fact that every planar graph of minimum degree three has an edge of weight $\leq 13$ [2] and avoid direct use of the discharging method.

For a fixed integer $b$, a vertex is called $b$-big if its degree is at least $b$, and $b$-small if the degree is less than $b$. If $b$ is known from the context, we drop the $b$-prefix and call the vertex big or small, respectively. If $x$ and $y$ is a pair of adjacent $b$-small vertices in a graph, we call the edge $xy$ $b$-light, or just light if $b$ is known from context.

By $P_k$ we denote the path on $k$ vertices, we call it also a $k$-path in order to emphasize that it is a copy of $P_k$ in some graph. Similarly, we define a $k$-cycle as a cycle of $k$ vertices in some graph and a $k$-vertex as a vertex of degree $k$.

2 Long and Heavy Paths in $\mathcal{P}_2$

In this section we study the structure of the graphs from $\mathcal{P}_2$ which contain $P_k$ as a subgraph and each such $k$-path contains at most two $b$-small vertices. Denote by $\mathcal{P}_2(b, k)$ this subclass of graphs of $\mathcal{P}_2$. Notice that if $a \geq b$, then $\mathcal{P}_2(a, k) \subseteq \mathcal{P}_2(b, k)$. Thus, if $b_1, b_2, b_3, \ldots$ is an increasing infinite sequence of integers and $\mathcal{P}_2(b_i, k) \neq \emptyset$ for each $i \geq 1$, then $\mathcal{P}_2(b_i, k) \neq \emptyset$ for each integer $b_i \geq 1$.

Lemma 1 Let $k$ and $b$ be two integers such that $b \geq k \geq 3$. Suppose that $G$ is a graph from $\mathcal{P}_2(b, k)$. Then $G$ contains a $(k + 1)$-path. Furthermore, if $G$ contains a $k$-path $P$ with two $b$-small vertices $s_1$ and $s_2$ such that the distance between $s_1$ and $s_2$ in $P$ is at most $k - 2$, then $G$ also contains a $(k + 1)$-path $P'$ such that $s_1$ and $s_2$ belong to $P'$, their distance in $P'$ is the same as in $P$ and one of vertices $s_1$ and $s_2$ is an end vertex of $P'$.

Proof. Let $P = x_1x_2\cdots x_k$ be a $k$-path in $G$. If $x_1$ or $x_k$ is big, then it has a neighbour outside of $P$, so we can extend $P$ to a path on $k + 1$ vertices. The same argument holds if $x_1$, $x_k$ are small and some of them has a neighbour outside of the path. So we may assume that both $x_1$ and $x_k$ are small and all their neighbours are on the path. Let $x_i$ be one of the neighbours of $x_1$ distinct from $x_2$. Such a vertex exist since $G \in \mathcal{P}_2$. Then $x_{i-1}$ must be big,
otherwise \( P \) would contain three small vertices. Therefore it has a neighbour \( w \) outside of \( P \). Then \( x_k x_{k-1} \cdots x_1 x_2 \cdots x_{i-1} x_i \) is a \((k + 1)\)-path in \( G \).

Now consider the case that \( P \) contains two small vertices \( x_i \) and \( x_j \) in distance at most \( k - 2 \). We may assume \( x_k \) is big and \( i < j \). We first construct a \( k \)-path \( P' = x'_1 x'_2 \cdots x'_k \) such that \( x'_1 = x_i \) and \( x'_{j-1+i} = x_j \). If \( i = 1 \), we set \( P' = P \). Otherwise let \( v \) be a neighbour of the big vertex \( x_k \) such that \( v \) does not belong to \( P \). We consider a \( k \)-path \( P'' = x_2 \cdots x_{k-1} x_k v \) instead of \( P \). The distance of \( x_i \) and \( x_j \) in \( P'' \) is unchanged and \( x_i \) is closer to start of \( P \), so after a finite number of repetitions of this construction we construct the path \( P'' \) as desired.

Vertex \( x'_k \) is big and therefore it has a neighbour \( v \) outside of \( P \). The \( k \)-path \( P' \) can then be extended to a \((k + 1)\)-path \( x'_1 x'_2 \cdots x'_k v \) that satisfies the conditions of the lemma. \( \square \)

For an integer \( b \), let \( l_b(G) \) be the number of \( b \)-light edges in \( G \). We define the following ordering: \( G_1 \prec_b G_2 \) if and only if either \( l_b(G_1) < l_b(G_2) \), or \( l_b(G_1) = l_b(G_2) \) and \( |E(G_1)| < |E(G_2)| \). We denote by \( \mathcal{P}_2'(b, k) \) the set of all minimal graphs of \( \mathcal{P}_2(b, k) \) in this ordering. Obviously, \( \mathcal{P}_2'(b, k) \) is non-empty if and only if \( \mathcal{P}_2(b, k) \) is non-empty. The following lemma shows that some configurations do not appear in the graphs of \( \mathcal{P}_2'(b, k) \).

**Lemma 2** Suppose that \( G \in \mathcal{P}_2'(b, k) \), with \( b \geq k > 4 \). Then \( G \) does not contain any of the following configurations as a subgraph:

(C1) Two adjacent small vertices \( u \) and \( v \) such that the degree of each of them is at least 3.

(C2) A vertex \( v \) of degree 2 adjacent to two nonadjacent vertices \( x \) and \( y \).

(C3) Two adjacent vertices \( x \) and \( y \) of degree 2 with a common small neighbour \( v \) of degree at least 4.

(C4) Two adjacent vertices \( x \) and \( y \) of degree 2 with a common small neighbour \( v \) of degree 3.

(C5) Vertices \( v_1 \) and \( v_2 \) of degree 2 with common adjacent neighbours \( x \) and \( y \) such that \( x \) is big and \( y \) is small.
Figure 1: Forbidden configurations and their reductions.

Proof. Suppose that $G$ contains some of these configurations. We construct a graph $G'$ such that $G' \in \mathcal{P}_2(b, k)$ and $G' \prec_b G$. Thus $G'$ will contradict the minimality of $G$. Fig. 2 illustrates the construction of $G'$ from $G$ for each of these configurations (dotted lines present that vertices are non-adjacent and labels $B$, $S$ that corresponding vertices are big or small, respectively). Consider each of the above configurations separately:

(C1) In the first configuration we remove the edge $e = uv$. This produces the planar graph $G'$ with minimum degree at least 2 and with smaller number of light edges than $G$. Thus $G' \prec_b G$.

We only have to show that $G' \in \mathcal{P}_2(b, k)$. A vertex is big in $G'$ if and only if it is big in $G$, and since $G'$ is a subgraph of $G$, the graph $G'$ does not contain any $k$-path with more than two small vertices. The graph $G$ must have at least one $k$-path, since if $P$ is a $k$-path in $G$ using the edge $e$, then according to Lemma 1, $G$ also contains a $(k + 1)$-path $Q$ such that $u$ and $v$ are the first two vertices of $Q$. Say $u$ is the first one. Then $Q - u$ is a $k$-path in $G'$. Therefore, $G'$ belongs to $\mathcal{P}_2(b, k)$.

(C2) In this configuration, we remove $v$ and add an edge $xy$ in order to obtain $G'$. This could create a new light edge $xy$, but then both $x$ and $y$ are small and so we remove two light edges $vx$ and $vy$. Therefore we
do not increase the number of light edges and we always decrease by
one the number of edges. Thus, \( G' \prec_b G \).

We have to show that \( G' \in \mathcal{P}_2(b, k) \). Obviously, \( G' \) has minimum de-
gree at least 2. We claim that the graph \( G' \) contains a \( k \)-path, since
according to Lemma 1, a \( k \)-path \( Q \) is a subgraph of \( G \) and the corre-
spanding path in \( G' \) is of length \( k \) or \( k + 1 \) (depending on whether \( v \)
belongs to \( Q \) or not). We also could not create a new \( k \)-path containing
more than 2 small vertices. Suppose \( P \) is such a path. Observe that
\( P \) contains the edge \( xy \). In \( G \) there is a corresponding path \( P' \) where
the edge \( xy \) is replaced by path \( xvy \). Thus, \( P' \) is a \((k + 1)\)-path with
at least 4 small vertices, and so \( P' \) contains a subpath of length \( k \) with
at least 3 small vertices.

\( \text{(C3)} \) In the third configuration we remove the vertices \( x \) and \( y \) to obtain \( G' \).
This removes three light edges, and so \( G' \prec_b G \).

If \( P \) is a \( k \)-path in \( G \) that does not occur in \( G' \), then \( P \) contains \( v \) and
one of \( x \) and \( y \). We may assume that \( P = xvu_1 \cdots u_{k-2} \). But then
\( P' = yxvu_1 \cdots u_{k-3} \) would be a \( k \)-path containing three small vertices
in \( G \), which is a contradiction. Therefore all \( k \)-paths in \( G \) also exist in
\( G' \) and it is now easy to see that graph \( G' \) belongs to \( \mathcal{P}_2(b, k) \).

\( \text{(C4)} \) Let \( u \) be the third neighbour of \( v \). We remove \( v \) from \( G \) and add edges
\( xu \) and \( yu \) in order to obtain \( G' \). This operation reduces the number
of light edges by at least one. Thus \( G' \prec_b G \) and \( G' \in \mathcal{P}_2 \).

Similarly to the configuration \( \text{(C3)} \), we observe that no \( k \)-path of \( G \\
uses a vertex from \{v, x, y\} \), since otherwise we would find a \( k \)-path
in \( G \) using three small vertices \( v, x \) and \( y \). Therefore we preserve all
\( k \)-paths of \( G \) by the described operation and so \( G' \) contains a \( k \)-path.

Note that we have increased the degree of \( u \) by one in this operation, so
\( u \) is big in \( G' \). Let \( P = v_1v_2 \cdots v_k \) be a \( k \)-path of \( G' \). If \( P \) occurs in \( G \\
it obviously has at most two small vertices. Otherwise we may assume
\( v_1 = x \) and \( v_2 \in \{y, u\} \). Then either \( vuv_3 \cdots v_k \) or \( xuvv_4 \cdots v_k \) is a
\( k \)-path in \( G \) containing the same number of small vertices. Therefore
\( G' \in \mathcal{P}_2(b, k) \).

\( \text{(C5)} \) If \( k = 5 \), consider the path \( v_1yv_2x \). It contains 3 small vertices and it
can be extended to a 5-path, since \( x \) is big. This is a contradiction.
If $k \geq 6$, we add $b - 2$ new vertices $v_3, \ldots, v_b$ and connect them to both $x$ and $y$ to obtain $G'$. Let $S = \{v_1, v_2, \ldots, v_b\}$. Since $y$ is big in $G'$, the edges $v_1y$ and $v_2y$ are not light in $G'$. Thus the number of light edges is reduced by at least 2, and hence $G' \prec_b G$.

In the construction of $G'$, we have not removed any path in $G$, so it is sufficient to argue that we could not create a $k$-path with at least three small vertices. Let $P$ be a $k$-path of $G'$. Then $P$ cannot use three of the vertices of $S$, since it would have length at most 5. If $P$ uses at most two vertices of $S$, we may assume these vertices belong to $\{v_1, v_2\}$ and then $P$ also exists in $G$. Since we have not decreased degree of any vertex, $P$ contains at most two small vertices in $G'$. Thus, $G' \in \mathcal{P}_2(b, k)$.

\[\square\]

If $xyz$ is a 3-cycle in a graph, $x$ and $y$ have degree 2 and $z$ is big, then we call the subgraph induced by these vertices a 3-booster. Similarly, if $xyuz$ is a 4-cycle in a graph, $x$ and $y$ have degree 2, $w$ and $z$ are big, we call the subgraph induced by these four vertices a 4-booster.

![Figure 2: Illustration of a 3-booster and a 4-booster.](image)

**Lemma 3** Let $k > 4$ be an integer. Suppose that for each integer $b \geq k$ the class $\mathcal{P}_2^*(b, k)$ is nonempty. Then for each integer $b \geq k$ there exists a planar graph $G_1(b, k) \in \mathcal{P}_2(b, k)$ satisfying the property:

(P1) If $x$ and $y$ are adjacent $b$-small vertices in $G_1(b, k)$, then $x$ and $y$ are of degree 2 and have a common $b$-big neighbour (i.e. $x$ and $y$ are a part of a 3-booster).

**Proof.** Let us fix one such $b$ and let $G' \in \mathcal{P}_2^*(2b, k)$. Due to Lemma 2:
- Graph $G'$ does not contain two adjacent $2b$-small vertices of degree at least three.

- If $G'$ contains two adjacent vertices $x$ and $y$ of degree 2, then they have a common neighbour that is $2b$-big (otherwise, $G'$ contains one of the configurations (C2), (C3) or (C4)).

- If $x$ and $y$ are adjacent, $x$ has degree 2, $y$ is a $2b$-small vertex of degree at least 3, then they have a common neighbour $z$ that is $2b$-big. Moreover, there is no 2-vertex distinct from $x$, which is a common neighbour of both $y$ and $z$.

We construct a graph $G = G_1(b, k)$ from $G'$ in the following way: For each pair of vertices $x$ and $y$ such that $x$ and $y$ are adjacent in $G'$, $x$ is $2b$-big, $2 < \text{deg}(y)$ and there exists a 2-vertex $v$ adjacent to $x$ and $y$ but not belonging to a 4-booster, we remove the vertex $v$.

A vertex with degree $d$ in $G'$ has a degree at least $\left\lceil \frac{d}{2} \right\rceil$ in $G$. Otherwise if $q$ is such a vertex and $v$ a 2-vertex adjacent to $q$ that is removed, the other neighbour $r$ of $v$ is adjacent to $q$ and $r$ and $q$ have no other common neighbour of degree 2 due to the restrictions on the structure of $G'$ described above. Consequently a vertex that has degree at least $2b$ in $G'$ has a degree at least $b$ in $G$.

In order to conclude the proof, let us check that graph $G$ satisfies conditions of the lemma:

- **Graph $G$ has a minimum degree at least 2.** The vertices incident to the removed edges had degree at least 3 in $G'$, so their degrees are at least $\left\lceil \frac{3}{2} \right\rceil = 2$ in $G$.

- **Each $k$-path in $G$ contains at most two $b$-small vertices.** Suppose $P$ is a $k$-path in $G$ with more than two $b$-small vertices. In $G'$, the path $P$ contains at most two $2b$-small vertices, therefore there must be a vertex $q$ on $P$ that has degree smaller than $b$ in $G$, and degree at least $2b$ in $G'$, which is a contradiction.

- **Graph $G$ contains a $k$-path.** Let $S = V(G') \setminus V(G)$ be the set of removed vertices. Let us arbitrarily order the elements of $S$ into a sequence $s_1, s_2, \ldots, s_m$. Thus we obtain a sequence of graphs $G' = G_0, G_1, \ldots, G_m = G$ where $G_{i+1} = G_i - s_{i+1}$. We prove by induction
on \( i \) that each \( G_i \) belongs to \( \mathcal{P}_2(b, k) \). By the above case we know that \( G_i \) does not contain \( k \)-path with more than two \( b \)-small vertices, therefore it suffices to prove that there is a \( k \)-path in \( G \).

We proceed by induction. For \( G_0 \) the statement is true. Let us assume that \( G_i \) contains a \( k \)-path. We will show that \( G_{i+1} \) contains a \( k \)-path as well. Note that the assumptions of Lemma 1 are preserved (for big vertices being those that have degree at least \( b \)). Therefore there also exists a \( (k + 1) \)-path \( P \) in \( G_i \). If \( s_{i+1} \) is an endvertex of \( P \), the path \( P - s_{i+1} \) is a \( k \)-path of \( G_{i+1} \). If \( s_{i+1} \) is some of inner vertices of \( P \), we know that the two adjacent neighbours \( x \) and \( y \) of \( s_{i+1} \) belong to \( P \) and therefore \( P - s_{i+1} \) together with the edge \( xy \) is a \( k \)-path in \( G_{i+1} \).

- **Graph \( G \) satisfies (P1).** Suppose \( x \) and \( y \) are adjacent \( b \)-small vertices in \( G \). No vertex with degree at least \( 2b \) in \( G' \) has degree less than \( b \) in \( G \). This means that both \( x \) and \( y \) are also \( 2b \)-small in \( G' \). Due to properties of \( G' \) one of \( x \) and \( y \), say \( x \), has degree 2 in \( G' \). If degree of \( y \) in \( G' \) is at least three, then \( x \) and \( y \) have a common \( 2b \)-big neighbour in \( G' \) and \( x \) should have been removed during construction of \( G \). Otherwise degree of \( y \) in \( G' \) is 2 as well and \( x \) and \( y \) have a common neighbour \( z \) that is \( 2b \)-big in \( G' \), and consequently \( z \) is \( b \)-big in \( G \). Therefore vertices \( x \), \( y \) and \( z \) induce a 3-booster in \( G \).

\[ \square \]

**Lemma 4** Let \( k > 4 \) be an integer. Suppose that for each \( b \geq k \) the class \( \mathcal{P}_2^b(b, k) \) is nonempty. Then for each integer \( b \geq k \) there exists a planar graph \( G_2(b, k) \in \mathcal{P}_2(b, k) \) satisfying the property (P1) of Lemma 3 and the property (P2) If \( x \) is a vertex of degree two in \( G \), then it is part of either a 3-booster or a 4-booster.

**Proof.** Let us fix one such \( b \). Let \( G' = G_1(2b, k) \) be a graph obtained by Lemma 3.

We construct \( G = G_2(b, k) \) in the following way: First we create a graph \( G'' \) by suppressing one by one 2-vertices of \( G' \) whose neighbours are not adjacent. Then we construct a graph \( G \) by removing each 2-vertex of \( G'' \) which does not belong to a 4-booster and whose neighbours are \( 2b \)-big.

Using the same argumentation as in the proof of the second case of Lemma 2, one can show that \( G'' \) belongs to \( \mathcal{P}_2(b, k) \) and that it has the property (P1).
Additionally, any vertex of degree two in $G''$ is either part of a 3-booster or a 4-booster, or its neighbours are two adjacent big vertices.

Similarly as in the proof of Lemma 3, one can show that $G \in P_3(b, k)$ and that it has the property (P1). Let us show that $G$ satisfies the property (P2) as well. This will finish the proof of the lemma.

Suppose $x$ has degree 2 in $G$ and it does not belong to a 3-booster. Then its two neighbours $u$ and $v$ are $b$-big. We need to show that $x$ belongs to a 4-booster. Suppose this is not the case. The degree of $x$ in $G''$ is 2, since we did not change degree of any vertex that was not $2b$-big in the construction of $G$. Due to suppressing the vertices of degree two in $G'$, we know that $u$ and $v$ must be adjacent in $G''$. Since they are adjacent to a 2-vertex in $G''$, vertices $u$ and $v$ are $2b$-big in $G''$. Therefore $x$ should have been removed during construction of $G$, which is a contradiction.

Lemma 5 Let $k > 4$ be an integer. Suppose that for each $b \geq k$ the class $P_3(b, k)$ is nonempty. Then for each integer $b \geq k$ there exists a planar graph $G'_3(b, k) \in P_2(b, k)$ satisfying the property (P1) of Lemma 3, the property (P2) of Lemma 4 and the property

(P3) No vertex of $G$ belongs to both a 3-booster and a 4-booster.

Proof. Let us fix one such $b$. By Lemma 4 we may assume $G' = G_2(b, k)$ exists. Only a big vertex of $G'$ can belong to both a 3-booster and a 4-booster. Let us construct the graph $G = G_3(b, k)$ in the following way: For each vertex $v$ in $G'$ such that it belongs both to a 3-booster with vertices $\{x_1, x_2, v\}$ and to a 4-booster with vertices $\{v, w, y_1, y_2\}$ (where $w$ is a big vertex), we perform the following operation: Let $R_v$ be a set of all vertices of degree 2 adjacent to $v$ and belonging to 3-boosters. Add a set $S_v$ of $|R_v|$ new vertices to $G$, and remove the vertices of $R_v$ from $G$. Join all vertices in $S_v$ to both $v$ and $w$.

Let $S'(v) = S(v) \cup \{y_1, y_2\}$. Let $f : V(G') \rightarrow V(G)$ be a bijection that maps each element of $R_v$ to an element of $S_v$, and it is an identity on vertices of $V(G) \cap V(G')$.

The graph $G$ obviously satisfies the properties (P1), (P2) and (P3), it is therefore sufficient to show that $G \in P_2(b, k)$ in order to finish the proof.

If $k = 5$, the configuration described in the construction cannot occur, otherwise we obtain a 5-path $x_1 x_2 y_1 z$ with at least three $b$-small vertices. So $G = G'$, and therefore $G$ belongs to $P_2(b, k)$.  

10
Now, we consider the case $k \geq 6$. Let $P = v_1v_2\cdots v_k$ be a $k$-path in $G'$ that does not exist in $G$. We may assume that no inner vertex of the path belongs to a 3-booster: If $v_1$ and $v_2$ are both 2-vertices, we can use Lemma 1 to get a $(k + 1)$-path $Q$ starting with $v_1v_2$ and so $Q - v_1$ is a $k$-path in $G$ such that all its inner vertices are big. Then only $v_1$ and $v_k$ could have been removed during the construction of $G$ and $f(v_1)f(v_2)\cdots f(v_k)$ is a $k$-path in $G$. Therefore $G$ contains a $k$-path.

It remains to show that $G$ does not contain a $k$-path with more than two small vertices. Let $P$ be a $k$-path in $G$. For each vertex $v$ the path $P$ uses at most two vertices of $S'_v$, since $k \geq 6$. So we may assume that $P$ only uses the vertices of $S'_v \setminus S_v$ and that therefore $P$ also occurs in $G'$. Then $P$ contains at most two small vertices in $G'$ and since the construction does not decrease the degree of any vertex, $P$ also contains at most two small vertices in $G$. □

3 Configurations of boosters

In this section we present lemmata showing that boosters are in some sense both frequent and rare in the graphs in $P_2(b, k)$ for $k \in \{7, 8, 9\}$. We use these results in the next section to determine the gravity of some paths.

Let $v$ be a big vertex of a graph $G \in P_2(b, k)$. We say that $v$ is boosted if $v$ either belongs to a booster, or if $v$ has $k + 1$ neighbours $v_0, \ldots, v_k$ such that for each $0 \leq i \leq k$ the vertex $v_i$ belongs to a 3-booster (notice that each $v_i$ must be big).

Madaras and Skrekovski [11] have proved that the gravity of $k$-paths in $P_2$ is at most $k - 2$. This means that for each $k$, there exists an integer $b'_0(k)$ such that for each $b \geq b'_0(k)$, any graph in $P_2$ that contains a $k$-path also contains a $k$-path with at least two $b$-small vertices. Let $b_0(k) = \max(k, b'_0(k))$.

**Lemma 6** For $k \geq 5$ and $b \geq b_0(k)$, let $G = G_3(b, k)$ be the graph constructed in Lemma 5. Suppose that for each $k$-path $P$ in $G$ with exactly two $b$-small vertices $s_1$ and $s_2$, the distance between $s_1$ and $s_2$ in $P$ is $k - 2$ or $k - 1$. Then $G$ contains a $b$-big vertex that is not boosted.

**Proof.** Since $b \geq b'_0(k)$, the graph $G$ contains a $k$-path $P = v_1\cdots v_k$ with two small vertices $s_1$ and $s_2$. Let $P$ be such that the distance between $s_1$ and $s_2$ in $P$ is the smallest possible. Vertices $s_1$ and $s_2$ have distance at least $k - 2$ in $P$ due to assumptions of the lemma, so we may assume that $s_1 = v_1$ and $s_2 \in \{v_{k-1}, v_k\}$.

11
If for any $2 < i < k$ the vertex $v_i$ is adjacent to $s_1$, then the graph $G$ contains the $k$-path $v_{i-1}v_{i-2} \cdots v_1v_{i+1} \cdots v_k$ where the distance from $s_1$ to $s_2$ is smaller by at least one than in $P$. Similarly if $s_1$ is adjacent to $v_k$, then the distance from $s_1$ to $s_2$ in the path $v_1v_kv_{k-1} \cdots v_2$ is at most two.

Therefore, the vertex $s_1$ cannot have a neighbour in $P$ distinct from $v_2$. Since the degree of $s_1$ is at least two, $s_1$ has a neighbour $v$ outside of $P$. The vertex $v$ cannot be small, otherwise $vv_1 \cdots v_{k-1}$ is a $k$-path in $G$ with two adjacent small vertices. Therefore $v$ is big. We show that $v$ cannot be boosted. If $v$ belongs to a booster, then let $x$ be the 2-vertex of the booster that is not in $P$. Then $xvv_1 \cdots v_{k-2}$ is a $k$-path where small vertices have distance two. Finally, if $v$ has $k + 1$ neighbours and each of them belongs to a 3-booster, then let $x$ be one of them that does not belong to $P$ and let $x_1$ and $x_2$ be the 2-vertices of the 3-booster of $x$. Then $x_1x_2xvv_1 \cdots v_{k-4}$ is a $k$-path in $G$ with three small vertices, a contradiction. \[\square\]

**Lemma 7** For $k \in \{7, 8, 9\}$ and $b \geq k$, let $G = G_3(b, k)$ be the graph constructed in Lemma 5. Suppose that $G$ contains a $k$-path $P$ in $G$ with two $b$-small vertices $s_1$ and $s_2$ such that the distance between $s_1$ and $s_2$ in $P$ is at most $k - 3$. Then $G$ contains a $b$-big vertex $v$ that is not boosted.

**Proof.** Let $P = v_1 \cdots v_k$. By Lemma 1, we may assume that $v_1 = s_1$. Then $v_{k-1}$ is big. The vertex $v_{k-1}$ does not belong to a booster, otherwise it has a neighbour $x$ of degree two distinct from $v_1$ and so $v_1 \cdots v_{k-1}x$ is a $k$-path in $G$ with three small vertices. Therefore, if $v_{k-1}$ is boosted, it must have a neighbour distinct from $v_1, \ldots, v_{k-2}$ which belongs to a 3-booster. We may assume that $v_k$ is such a neighbour and let $v_{k+1}$ and $v_{k+2}$ be the 2-vertices of its 3-booster. Then the remaining small vertex of $P$ must be $v_2$ (otherwise the $k$-path $v_3 \cdots v_{k+2}$ contains three small vertices), and therefore $\{v_1, v_2, v_3\}$ must induce a 3-booster in $G$ due to property (P1).

Now, let us argue that $v_5$ cannot be boosted. The vertex $v_5$ cannot belong to a 3-booster, since if $x_1$ and $x_2$ are the 2-vertices of such a 3-booster, then $x_1x_2v_5v_6 \cdots v_{k+2}$ is a $k$-path with four small vertices. The vertex $v_5$ also cannot have a neighbour outside $P$ belonging to a 3-booster, since if $x$ is such a neighbour and $y$ is a small vertex of the 3-booster, then $xyv_5v_6 \cdots v_{k+2}$ is a $k$-path with three small vertices.

So if $v_5$ is boosted, it belongs to a 4-booster. Let $x$ be the other big vertex and $y$ is a 2-vertex of this 4-booster. By property (P3), $x \neq v_k$. Then $x \in \{v_6, \ldots, v_{k-1}\}$, otherwise $xyv_5v_6 \cdots v_{k+2}$ is a $k$-path with three small
vertices. But \(v_{k-1}\) does not belong to any booster as argued before, and if
\(x = v_i\) for \(i \in \{6, 7\}\), then \(v_1 v_2 v_3 v_4 v_5 v_6 \cdots v_{k+3-7}\) is a \(k\)-path containing three
small vertices. Since \(k \leq 9\), we considered all possible values of \(i\).

By putting Lemma 6 and Lemma 7 together, we obtain

**Corollary 8** For \(k \in \{7, 8, 9\}\) and \(b \geq b_0(k)\), let \(G = G_3(b, k)\) be the graph
constructed in Lemma 5. Then \(G\) contains a \(b\)-big vertex \(v\) that is not boosted.

We further study the structure of the neighbourhood of a vertex \(v\) that
is not boosted. We are especially interested in the following configurations,
where \(z\) is a neighbour of the vertex \(v\) from Corollary 8:

(B1) Vertex \(z\) is big and belongs to a 3-booster.

(B2) Vertex \(z\) is big and belongs to a 4-booster.

(B3) Vertex \(z\) is big and has \(k + 1\) big neighbours, where each belongs to
some 3-booster.

(B4) Vertex \(z\) is small and adjacent to a big vertex \(w\) that belongs to some
3-booster.

(B5) Vertex \(z\) is small and adjacent to a big vertex \(w\) that belongs to some
4-booster.

(B6) Vertex \(z\) is small and adjacent to a big vertex \(w\) having \(k + 1\) big
neighbours, where each belongs to some 3-booster.

Notice that \(w \neq v\) in configurations (B4) – (B6), since \(v\) is not boosted.
Let us call the edge \(vz\) in each of these configurations important. Let us em-
phasize that in case of the configuration (B2), if \(v\) is also adjacent to the other
big vertex of the 4-booster, we consider this as two separate instances of the
configuration (B2), and that each of these instances has just one important
edge.

**Lemma 9** For \(k \in \{7, 8, 9\}\) and \(b \geq k\), let \(G = G_3(b, k)\) be the graph con-
structed in Lemma 5. Let \(v\) be a big vertex of \(G\) that is not boosted. Then \(v\)
is incident with at most

(a) \(k\) important edges of instances of configuration (B1);
(b) 4 important edges of instances of configuration (B2);
(c) 1 important edge of an instance of configuration (B3);
(d) 1 important edge of an instance of configuration (B4);
(e) 3 important edges of instances of configuration (B5);
(f) 3 important edges of instances of configuration (B6).

Proof. Note that such vertex $v$ exists due to Corollary 8. Let us handle the configurations (B1)–(B6) one by one:

(B1) If $v$ is incident with $k + 1$ important edges of instances of configuration (B1), then $v$ is boosted.

(B2) Suppose that $v$ is incident with 5 important edges of instances of configuration (B2). Then no two 4-boosters in these instances are disjoint, otherwise $G$ contains a $k$-path with at least three small vertices. Using the pigeonhole principle, we observe that among these 5 instances there must be three of them that share only the big vertex of the 4-boosters that is not incident to the important edge of each of the instances (see Figure 3). Note that this configuration contains a $k$-path with at least three vertices of degree two for each $k \in \{7, 8, 9\}$, which is a contradiction.

![Figure 3: Forbidden path in case (B2)](image)

(B3) If $k = 7$, the configuration (B3) contains a 7-path with four small vertices. So we may assume $k = 8$ or 9.
Suppose that \( v \) is incident with 2 important edges of instances of configuration (B3). Let \( z_1 \) and \( z_2 \) be the vertices of these edges distinct from \( v \). Then we can find two distinct big vertices \( x_1 \) and \( x_2 \) such that each of them is distinct from \( z_1 \) and \( z_2 \), both \( x_1 \) and \( x_2 \) belong to 3-boosters and \( x_1z_1 \) and \( x_2z_2 \) are edges of \( G \). This configuration however contains a \( k \)-path with at least three small vertices.

(B4) Let \( v \) be incident with 2 important edges of instances of configuration (B4). Let \( z_1 \) and \( z_2 \) be the small vertices incident with the important edges of the configurations. By property (P1), vertices \( z_1 \) and \( z_2 \) are not adjacent. The two configurations corresponding to them cannot be disjoint, otherwise they contain a \( k \)-path with at least three small vertices. More precisely there is a big vertex \( w \) adjacent to both \( z_1 \) and \( z_2 \) such that \( w \) belongs to a 3-booster. Let \( w_1 \) and \( w_2 \) be the small vertices of that booster. Since \( v \) is not boosted, it is not adjacent to vertex of degree two, and therefore degree of \( z_2 \) must be at least three. Consequently the vertex \( z_2 \) has other big neighbour \( x \) distinct from \( v \) and \( w \). Since \( x \) is big, it has a neighbour not in \( \{z_1, z_2, v, w\} \), let this neighbour be \( y \).

Then \( Q = w_1w_2wz_1vz_2xy \) is an 8-path containing at least four small vertices, which cannot occur unless \( k = 9 \). If \( y \) would have a neighbour outside \( Q \), we could extend \( Q \) to a 9-path. Consequently, \( y \) must be small. Besides \( x \), the only neighbours of \( y \) could be \( v \) and \( w \) due to property (P1). If \( y \) is not adjacent to both \( v \) and \( w \), vertex \( y \) has degree 2. Then the vertex \( y \) belongs to a 4-booster due to property (P2), and consequently \( v \) or \( w \) belongs to this 4-booster. But this cannot occur: the vertex \( v \) is not boosted, and by property (P3), the vertex \( w \) cannot belong to both a 3-booster and a 4-booster. Therefore \( y \) must be adjacent to both \( v \) and \( w \).

Since \( y \) was an arbitrary neighbour of \( x \) distinct from \( \{z_1, z_2, v, w\} \), we observe that any neighbour of \( x \) not in this set must be adjacent to both \( v \) and \( w \). But due to planarity of \( G \) the vertex \( x \) can have at most two such neighbours, otherwise \( G \) contains \( K_{3,3} \) as a subgraph. This means that the degree of \( x \) is at most 6, which is a contradiction with the fact that \( x \) is a big vertex.

(B5) Suppose that \( v \) is incident with 4 important edges of instances of configuration (B5). Let \( z_1 \), \( z_2 \), \( z_3 \) and \( z_4 \) be the small vertices incident
with the important edges of the configurations. Since $v$ is not boosted, property (P2) implies that the degree of each $z_i$ is at least three. Let $w$ be the big vertex adjacent to $z_1$ belonging to a 4-booster, $w_1$ and $w_2$ the small vertices of this booster and $y$ the remaining big vertex of this booster. Then $Q_i = w_1yw_2wz_1vz_i$ for $i \in \{2, 3, 4\}$ is a 7-path with four small vertices, so this cannot occur if $k = 7$. If $z_i$ has a neighbour $x$ distinct from $w$, $y$ and $v$, then the vertex $x$ must be big due to property (P1). Then $x$ has other neighbour outside $Q_i$ and together they extend $Q_i$ to a 9-path. Therefore $z_i$ has no such neighbour. Since degree of $z_i$ is at least three, $z_i$ must be adjacent to both $w$ and $y$. But then $K_{3,3}$ is a subgraph of $G$, which contradicts the planarity of $G$.

(B6) If $k = 7$ the configuration (B6) contains a 7-path with four small vertices, so we may assume $k = 8$ or 9.

Let $v$ be incident with 4 important edges of instances of configuration (B6). Let $z_1$, $z_2$, $z_3$ and $z_4$ be the vertices of these important edges distinct from $v$. Similarly as in the previous case we see that the degree of each $z_i$ is at least three. Let $w$ be the big vertex of the configuration adjacent to $z_1$ and let $y$ be one of the big vertices adjacent to $w$ that belongs to a 3-booster. Let $y_1$ and $y_2$ be the small vertices of this 3-booster.

The 7-path $Q_i = y_1y_2ywz_1vz_i$ (for $i \in \{2, 3, 4\}$) contains four small vertices. If $z_i$ has a neighbour $x$ distinct from $v$, $w$ and $y$, then vertex $x$ must be big, and therefore $x$ together with one of its neighbours would extend $Q_i$ to a 9-path. Since degree of $z_i$ is at least three, one can easily see that $z_i$ must be adjacent to both $w$ and $y$. But this is again a contradiction with the planarity of $G$.

\[\square\]

4 Gravity of paths in $\mathcal{P}_2$

We are now ready to determine the gravity of $k$-paths in $\mathcal{P}_2$ for $k \in \{5, 7, 8, 9\}$.

**Theorem 10** Gravity of $P_5$ in the class of planar graphs with minimal degree 2 is 2.
Proof. Due to [11], gravity of \( P_3 \) in \( \mathcal{P}_2 \) is either two or three. Suppose for the sake of contradiction that gravity of \( P_3 \) is three. This means for infinitely many integers \( b \), the class \( \mathcal{P}_2(b, 5) \) is non-empty. So by the remark at the beginning of Section 2, one concludes that the class \( \mathcal{P}_2(b, 5) \) is non-empty for all integers \( b \geq 1 \). Let us consider the graph \( H = G_3(b, 5) \) constructed in Lemma 5, for sufficiently large \( b \) (at least 13).

Each big vertex \( v \) of \( H \) belongs to at most one booster, otherwise one can find a 5-path with three small vertices. By property (P1) a small vertex of degree at least three has only big neighbours.

We construct a graph \( H' \) by removing all vertices of degree 2 from \( H \). Thus, big vertices lose at most two neighbours, the degree of remaining small vertices is unchanged. Therefore the graph \( H' \) has minimum degree three and each its edge contains at least one vertex of degree at least \( b - 2 \geq 11 \). But due to [2] each planar graph with minimum degree three contains an edge \( e \) such that sum of degrees of the vertices incident with \( e \) is at most 13, which is a contradiction. \[ \square \]

**Theorem 11** Gravity of \( P_k \) in the class of planar graphs with minimal degree 2 is \( k - 3 \) for \( k \in \{7, 8, 9\} \).

Proof. Let \( k \in \{7, 8, 9\} \). Due to [11] gravity of \( k \)-paths in \( \mathcal{P}_2 \) is either \( k - 2 \) or \( k - 3 \). Suppose for the sake of contradiction that gravity of \( P_k \) is \( k - 2 \). Similarly as in the above theorem, one can conclude that \( \mathcal{P}_3(b, k) \) is nonempty for all integers \( b \geq 1 \). Let us consider a graph \( H = G_3(b, k) \) constructed in Lemma 5, for \( b \) sufficiently large (at least \( \max(b_0(k), k + 23) \)). Now, we construct a graph \( H' \) by removing all boosted vertices and all small vertices that are adjacent to boosted vertices from \( H \) (note that by property (P2) every 2-vertex is removed).

By the choice of \( b \), the graph \( H \) has a \( k \)-path which contains precisely two small vertices. So, Corollary 8 implies that \( H \) contains a big vertex that is not boosted, and therefore \( H' \) is nonempty. If \( v \) is a vertex of \( H' \) that is big in \( H \), then Lemma 9 implies that vertex \( v \) has degree at least \( k + 23 - (k + 12) = 11 \) in \( H' \). If \( v \) is a vertex of \( H' \) that is small in \( H \), then no neighbour of \( v \) was removed and \( v \) has degree at least three both in \( H \) and in \( H' \), since a small vertex of degree at least three has only big neighbours due to property (P1).

Therefore \( H' \) has minimum degree at least 3 and the sum of degrees of vertices of any of edges of \( H' \) is at least 14 (since there are no two small ad-
jacent vertices of degree at least three in $H$). This however is a contradiction with results of [2].

References


