THE BRYANT-FERRY-MIO-WEINBERGER CONSTRUCTION OF GENERALIZED MANIFOLDS

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CONSTRUCTION OF GENERALIZED MANIFOLDS

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Abstract. Following Bryant, Ferry, Mio and Weinberger we construct generalized manifolds as limits of controlled sequences \( \{X_i \xrightarrow{p_i} X_{i-1} \mid i = 1, 2, \ldots \} \) of controlled Poincaré spaces. The basic ingredient is the \( \varepsilon, \delta \)-surgery sequence recently proved by Pedersen, Quinn and Ranicki. Since one has to apply it not only in cases when the target is a manifold, but a controlled Poincaré complex, we explain that issue very roughly (Theorem 3.5). Specifically, it will be applied in the inductive step to construct the desired controlled homotopy equivalence \( p_{i+1} : X_{i+1} \to X_i \). Theorem 3.5 requires a sufficiently controlled Poincaré structure on \( X_i \) (over \( X_{i-1} \)). Our construction shows that this can be achieved. In fact, the Poincaré structure of \( X_i \) depends upon a homotopy equivalence used to glue two manifold pieces together (the rest is surgery theory leaving unaltered the Poincaré structure). It follows from the \( \varepsilon, \delta \)-surgery sequence (more precisely from the Wall realization part) that this homotopy equivalence is sufficiently well controlled. In §4 we give additional explanation why the limit space of the \( X_i \)’s has no resolution.

§ 1. Preliminaries

A generalized \( n \)-dimensional manifold \( X \) is characterized by the following two properties:

(i) \( X \) is a Euclidean neighborhood retract (ENR); and

(ii) \( X \) has the local homology (with integer coefficients) of the Euclidean \( n \)-space \( \mathbb{R}^n \), i.e.

\[
H_*(X, X \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).
\]

Since we deal here with locally compact separable metric spaces of finite (covering) dimension, ENR’s are the same as ANR’s.

Generalized manifolds are Poincaré spaces, in particular they have the Spivak normal fibrations \( \nu_X \). The total space of \( \nu_X \) is the boundary of a regular neighborhood \( N(X) \subset \mathbb{R}^L \) of an embedding \( X \subset \mathbb{R}^L \), for some large \( L \). One can assume that \( N(X) \) is a mapping cylinder neighborhood (see [Lam Corollary 11.2]).

The global Poincaré duality of Poincaré spaces does not imply the local homology condition (ii) above. The local homology condition can be understood as the
"controlled" global Poincaré duality (see [Qu83; p. 270], and [BFMW; Proposition 4.5]). More precisely, one has the following:

**Theorem 1.1.** Let $X$ be a compact ANR Poincaré duality space of finite (covering) dimension. Then $X$ is a generalized manifold if and only if for every $\delta > 0$, $X$ is a $\delta$–Poincaré space (over $X$).

The definition of the $\delta$–Poincaré property is given below. The following basic fact about homology manifolds was proved by Ferry and Pedersen (see [FP; Theorem 16.6]):

**Theorem 1.2.** Let $X$ be an ANR homology manifold. Then $\nu_X$ has a canonical TOP reduction.

This statement is equivalent to existence of degree one normal maps $f : M^n \to X$, where $M^n$ is a (closed) topological $n$–manifold, hence the structure set $\mathcal{S}_{\text{TOP}}(X)$ can be identified with $[X, G/\text{TOP}]$.

Let us denote the 4–periodic simply connected surgery spectrum by $\mathbb{L}$ and let $G/\text{TOP}$ be the connected covering of $\mathbb{L}$. There is a (canonical) map of spectra $\mathcal{G}/\text{TOP} \to \mathbb{L}$ given by the action of $G/\text{TOP}$ on $\mathbb{L}$.

If $M^n$ is a topological manifold there exists a fundamental class $[M]_\mathbb{L} \in H_n(M, \mathbb{L}^*)$, where $\mathbb{L}^*$ is the symmetric surgery spectrum (see [Ra; Chapters 13 and 16]).

**Theorem 1.3.** If $M^n$ is a closed oriented topological $n$–manifold, then the cap product with $[M]_\mathbb{L}$ defines a Poincaré duality of $\mathbb{L}$–(co–)homology

$$H^p(M, \mathbb{L}) \xrightarrow{\cong} H_{n-p}(M, \mathbb{L})$$

and $G/\text{TOP}$ (co–)homology

$$H^p(M, G/\text{TOP}) \xrightarrow{\cong} H_{n-p}(M, G/\text{TOP})$$

Since $H^0(M, \mathbb{L}) = [M, \mathbb{Z} \times G/\text{TOP}]$ and $H^0(M, G/\text{TOP}) = [M, G/\text{TOP}]$, one has $H_n(M, \mathbb{L}) = \mathbb{Z} \times H_n(M, G/\text{TOP})$ and the map $\mathcal{G}/\text{TOP} \to \mathbb{L}$ has the property that the image of $H_n(M, G/\text{TOP}) \to H_n(M, \mathbb{L}) = \mathbb{Z} \times H_n(M, G/\text{TOP})$ is $\{1\} \times H_n(M, G/\text{TOP})$ (see [Ra; Appendix C]). Moreover, the action of $\mathcal{H}^0(M, G/\text{TOP})$ on $H^0(M, \mathbb{L}) = \mathbb{Z} \times H^0(M, G/\text{TOP})$, induced by the action of $G/\text{TOP}$ on $\mathbb{L}$, preserves the $\mathbb{Z}$–sectors.

If $X$ is a generalized $n$–manifold we get similar results by using the fundamental class $f_*([M]_\mathbb{L}) = [X]_\mathbb{L} \in H_n(X, \mathbb{L}^*)$, where $f : M \to X$ is the canonical degree one normal map. So the composition map

$$\Theta : [X, G/\text{TOP}] \to H_n(X, G/\text{TOP}) \to H_n(X, \mathbb{L}) = \mathbb{Z} \times H_n(X, G/\text{TOP})$$

has the property that $\text{Im} \Theta$ belongs to a single $\mathbb{Z}$–sector, denoted by $I(X) \in \mathbb{Z}$.

The following is the fundamental result of Quinn on resolutions of generalized manifolds (see [Qu87]):
Theorem 1.4. Let $X$ be a generalized $n$–manifold, $n \geq 5$. Then $X$ has a resolution if and only if $I(X) = 1$.

Remark. The integer $I(X)$ is called the Quinn index of the generalized manifold $X$. Since the action of $G_{\text{TOP}}$ on $\mathbb{L}$ preserves the $\mathbb{Z}$–sectors, arbitrary degree one normal maps $g : N \to X$ can be used to calculate $I(X)$. Alternatively, we can define $I(X)$ using the fibration $G_{\text{TOP}} \to \mathbb{L} \to \mathbb{K}(\mathbb{Z}, 0)$, where $\mathbb{K}(\mathbb{Z}, \cdot)$ is the Eilenberg–MacLane spectrum, and define $I(X)$ as the image of (see [Ra; Chapter 25]):

$$\{ f : M \to X \} \in H_n(X, \mathbb{L}) \to H_n(X, \mathbb{K}(\mathbb{Z}, 0)) = H_n(X, \mathbb{Z}) = \mathbb{Z}.$$

We assume that $X$ is oriented. Therefore $I(X)$ is also defined for Poincaré complexes, as long as we have a degree one normal map $f : M \to X$, determining an element in $H_n(X, \mathbb{L})$. In this case $I(X)$ is not a local index. In fact, for generalized manifolds one has local $\mathbb{L}$–Poincaré duality using locally finite chains, hence we can define $I(\mathcal{U})$ for any open set $\mathcal{U} \subset X$. It is also easy to see that $I(\mathcal{U}) = I(X)$. On the algebraic side $I(X)$ is an invariant of the controlled Poincaré duality type (see [Ra; p. 283]).

§ 2. Constructing generalized manifolds from controlled sequences of Poincaré complexes

Beginning with a closed topological $n$–manifold $M^n$, $n \geq 5$, and $\sigma \in H_n(M, \mathbb{L})$, we shall construct a sequence of closed Poincaré duality spaces $X_0, X_1, X_2, \ldots$ and maps $p_i : X_i \to X_{i-1}, p_0 : X_0 \to M$.

We assume that $M$ is a PL manifold, or that $M$ has a cell structure. The $X_i$'s are built by gluing manifolds along boundaries with homotopy equivalences, and by doing some surgeries outside the singular sets. Hence all $X_i$'s have cell decompositions.

We can assume that the $X_i$'s lie in a (large enough) Euclidean space $\mathbb{R}^L$ which induces the metric on $X_i$. So the cell chain complex $C_\#(X_i)$ can be considered as a geometric chain complex over $X_{i-1}$ with respect to $p_i : X_i \to X_{i-1}$, i.e. the distance between two cells of $X_i$ over $X_{i-1}$ is the distance between the images of the centers of these two cells in $X_{i-1}$. Let us denote the distance function by $d$.

We now list five properties of the sequence $\{(X_i, p_i)\}_i$, including some definitions and comments. For each $i \geq 0$ we choose positive real numbers $\xi_i$ and $\eta_i$.

(i) $p_i : X_i \to X_{i-1}$ and $p_0 : X_0 \to M$ are $UV^1$–maps. This means that for every $\varepsilon > 0$ and for all diagrams

$$K_0 \xrightarrow{\alpha_0} X_i \cap \quad \xrightarrow{p_i} \quad K \xrightarrow{\alpha} X_{i-1}$$

$K$ a 2–complex, $K_0 \subset K$ a subcomplex and maps $\alpha_0, \alpha$, there is a map $\overline{\alpha}$ such that $\alpha |_{K_0} = \alpha_0$ and $d(p_i \circ \overline{\alpha}, \alpha) < \varepsilon$. (This is also called $UV^1(\varepsilon)$ property.)
(ii) $X_i$ is an $\eta_i$–Poincaré complex over $X_{i-1}$, i.e.
(a) all cells of $X_{i-1}$ have diameter $< \eta_i$ over $X_{i-1}$; and
(b) there is an $n$–cycle $c \in C_n(X_i)$ inducing an $\eta_i$–chain equivalence $c : C^\#(X_i) \to C^\#(X_i)$.

Equivalently, the diagonal $\Delta^\#(c) = \sum c' \otimes c'' \in C^\#(X) \otimes C^\#(X)$ has the property that $d(c', c'') < \eta$ for all tensor products appearing in $\Delta^\#(c)$.

(iii) $p_i : X_i \to X_{i-1}$ is an $\xi_i$–homotopy equivalence over $X_{i-2}$, i.e.
(a) there exist $p_i' : X_{i-1} \to X_i$ and homotopies $h_i : p_i' \circ p_i \simeq \text{Id}_{X_i}$,
(b) there exist $p_i' : X_{i-1} \to X_i$ and homotopies $h_i : p_i' \circ p_i \simeq \text{Id}_{X_i}$, such that the tracks $\{(p_i \circ p_i' \circ h_i)(x, t) : t \in [0, 1]\}$ and $\{(p_i \circ p_i' \circ h_i)(x, t) : t \in [0, 1]\}$ have diameter $< \xi_i$, for each $x \in X_i$ (resp. $x' \in X_i$). Note that $p_0$ need not be a homotopy equivalence.

(iv) There is a regular neighborhood $W_0 \subset \mathbb{R}^L$ of $X_0$ such that $X_i \subset W_0$, $i = 0, 1, ...$ and retractions $r_i : W_0 \to X_i$, satisfying $d(r_i, r_{i-1}) < \xi_i$ in $\mathbb{R}^L$.

(v) There are ”thin” regular neighborhoods $W_i \subset \mathbb{R}^L$, $\pi_i : W_i \to X_i$, with $W_i \subset W_{i-1}$ such that $W_{i-1} \setminus \bar{W}_i$ is an $\xi_i$–thin $h$–cobordism with respect to $r_i : W_0 \to X_i$.

Let $W = W_{i-1} \setminus \bar{W}_i$. Then there exist deformation retractions $r_t^0 : W \to \partial_0 W$ and $r_t^1 : W \to \partial_1 W$ with tracks of size $< \xi_i$ over $X_{i-1}$, i.e. the diameters of $\{(r_t \circ r_t^0)(w) : t \in [0, 1]\}$ and $\{(r_t \circ r_t^1)(w) : t \in [0, 1]\}$ are smaller than $\xi_i$. Moreover, we can choose $\eta_i$ and $\xi_i$ so that:
(a) $\sum \eta_i < \infty$; and
(b) $W_{i-1} \setminus \bar{W}_i$ has a $\delta_i$–product structure with $\sum \delta_i < \infty$, i.e. there is a homeomorphism $W = W_{i-1} \setminus \bar{W}_i \xrightarrow{H} \partial_0 W \times I$ satisfying $\text{diam}\{(r_t \circ H)(w, t) : t \in I\} < \delta_i$, for every $w \in \partial_0 W$.

The property (v)(b) above follows from the ”thin $h$–cobordism” theorem (see [Qu79]). One can assume that $\sum \xi_i < \infty$. Let $X = \cap_i W_i$. We are going to show that $X$ is a generalized manifold:

(1) The map $r = \lim r_i : W_0 \to X$ is well-defined and is a retraction, hence $X$ is an ANR.

(2) To show that $X$ is a generalized manifold we shall apply the next two theorems. They also imply Theorem 1.1 above. The first one is due to Daverman and Husch [DH], but it is already indicated in [Qu79] (see the remark after Theorem 3.3.2):

**Theorem 2.1.** Suppose that $M^n$ is a closed topological $n$–manifold, $B$ is an ANR, and $p : M \to B$ is proper and onto. Then $B$ is a generalized manifold, provided that $p$ is an approximate fibration.

Approximate fibrations are characterized by the property that for every $\varepsilon > 0$ and every diagram
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\[
\begin{array}{rcc}
K \times \{0\} & \xrightarrow{H_0} & M \\
\cap & \searrow \downarrow & p \\
K \times I & \xrightarrow{h} & B
\end{array}
\]

where \( K \) is a polyhedron, there exists a lifting \( H \) of \( h \) such that \( d(p \circ H, h) < \varepsilon \). Here \( d \) is a metric on \( B \). In other words, \( p : M \to B \) has the \( \varepsilon \)-homotopy lifting property for all \( \varepsilon > 0 \).

We apply Theorem 2.1 to the map \( \rho : \partial W_0 \to X \) defined as follows: Let \( \rho : W_0 \to X \) be the map which associates to \( w \in W_0 \) the endpoint \( \rho(x) \in X \) following the tracks defined by the "thin" product strucers of the \( h \)-cobordism when decomposing

\( W_0 = (W_0 \setminus \hat{W}_1) \cup (W_1 \setminus \hat{W}_2) \cup ... \)

The restriction to \( \partial W_0 \) will also be denoted by \( \rho \). By (v)(b) above the map \( \rho \) is well-defined and continuous. We will show that it is an \( \varepsilon \)-approximate fibration for all \( \varepsilon > 0 \).

The map \( \rho : W_0 \to X \) is the limit of maps \( \rho_i : W_0 \to X_i \), where \( \rho_i \) is the composition given by the tracks \( (W_0 \setminus \hat{W}_1) \cup (W_1 \setminus \hat{W}_2) \cup ... \cup (W_{i-1} \setminus \hat{W}_i) \) followed by \( \pi_i : W_i \to X_i \). The second theorem is Proposition 4.5 of [BFMW]:

**Theorem 2.2.** Given \( n \) and \( B \), there exist \( \varepsilon_0 > 0 \) and \( T > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \) the following holds: If \( X \xrightarrow{p} B \) is an \( \varepsilon \)-Poincaré complex with respect to the \( UV^1 \)-map \( p \) and \( W \subset \mathbb{R}^L \) is a regular neighborhood of \( X \subset \mathbb{R}^L \), i.e. \( \pi : W \to X \) is a neighborhood retraction, then \( \pi \big|_{\partial W} : \partial W \to X \) has the \( T\varepsilon \)-lifting property, provided that the codimension of \( X \) in \( \mathbb{R}^L \) is \( \geq 3 \).

This is applied as follows: Let \( B \subset \mathbb{R}^L \) be a (small) regular neighborhood of \( X \subset \mathbb{R}^L \). Hence \( X_k \subset W_k \subset B \) for sufficiently large \( k \). It follows by property (ii) that \( X_i \) is an \( \eta_i \)-Poincaré complex over \( X_i \xrightarrow{p_i} X_{i-1} \subset B \), hence (for \( i \) sufficiently large) we get the following:

**Corollary 2.3.** \( \rho_i : \partial W_0 \to X_i \) is a \( T\eta_i \)-approximate fibration over \( B \).

**Proof.** By the theorem above, \( \pi_i : \partial W_i \to X_i \) is a \( T\eta_i \)-approximate fibration over \( B \), hence so is \( \rho_i : \partial W_0 \cong \partial W_i \to X_i \). \( \square \)

It follows by construction that \( \lim X_i = X \subset B \), hence we have in the limit an approximate fibration \( \rho : \partial W_0 \to X \) over \( \text{Id} : X \to X \), i.e. \( X \) is a generalized manifold. We will show in \( \S 4 \) that \( I(X) \) is determined by the \( \mathbb{Z} \)-sector of \( \sigma \in H_n(M, \mathbb{L}) \).

\( \S 3. \) CONSTRUCTION OF THE SEQUENCE OF CONTROLLED POINCARÉ COMPLEXES

Before we begin with the construction we need more fundamental results about controlled surgery and approximations.
Part I. $\epsilon - \delta$ surgery theory.

We recall the main theorem of [PQR]. Let $B$ be a finite-dimensional compact ANR, and $N^n$ a compact $n$–manifold (possibly with nonempty boundary $\partial N$), $n \geq 4$. Then there exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exist $\delta > 0$ with the following property:

If $p : N \to B$ is a $UV^1(\delta)$ map, then there exists a controlled exact surgery sequence:

$$H_{n+1}(B, \mathbb{L}) \to S_{\varepsilon, \delta}(N, p) \to [N, \partial N; G/\text{TOP}, \ast] \xrightarrow{\Theta} H_n(B, \mathbb{L}).$$

The controlled structure set $S_{\varepsilon, \delta}(N, p)$ is defined as follows: Elements of $S_{\varepsilon, \delta}(N, g)$ are (equivalence) classes of $(M, g)$, where $M$ is an $n$–manifold, $g : M \to N$ is a $\delta$–homotopy equivalence over $B$ and $g|_{\partial M} : \partial M \to \partial N$ is a homeomorphism. The pair $(M, g)$ is related to $(M', g')$ if there is a homeomorphism $h : M \to M'$, such that the diagram

$$\begin{array}{ccc}
\partial M & \xrightarrow{h} & \partial M' \\
g \downarrow & & \downarrow g' \\
\partial N & & 
\end{array}$$

commutes, and $g' \circ h$ is $\varepsilon$–homotopic to $g$ over $B$. Since $\varepsilon$ is fixed, this relation is not transitive. It is part of the assertion that it is actually an equivalence relation. Then $S_{\varepsilon, \delta}(N, p)$ is the set of equivalence classes of pairs $(M, g)$.

As in the classical surgery theory, the map

$$H_{n+1}(B, \mathbb{L}) \to S_{\varepsilon, \delta}(N, p)$$

is the controlled realization of surgery obstructions, and

$$S_{\varepsilon, \delta}(N, p) \to [N, \partial N; G/\text{TOP}, \ast] \xrightarrow{\Theta} H_n(B, \mathbb{L})$$

is the actual (controlled) surgery part. The following discussion will show that (3.3) also holds for controlled Poincaré spaces (see Theorem 3.5 below). Moreover, $\delta$ is also of (arbitrary) small size, provided that such is also $\varepsilon$.

To see this we will go through some of the main points of the proof of Theorem 1 from [PQR]. For $\eta', \eta > 0$ we denote by $L_n(B, \mathbb{Z}, \eta', \eta)$ the set of highly $\eta$–connected $n$–dimensional quadratic Poincaré complexes modulo highly $\eta'$–connected algebraic cobordisms. Then there is a well-defined obstruction map

$$\Theta_\eta : [N, G/\text{TOP}] \to L_n(B, \mathbb{Z}, \eta', \eta)$$

(for simplicity we shall assume that $\partial N = \emptyset$). If $(f, b) : M^n \to N^n$ is a degree one normal map one can do controlled surgery to obtain a highly $\eta$–connected normal map $(f', b') : M'^n \to N^n$ over $B$. If $N^n$ is a manifold this can be done for every
\( \eta > 0 \). If \( N^m \) is a Poincaré complex, it has to be \( \eta \)-controlled over \( B \). By Theorem 1.1 above this holds in particular for generalized manifolds.

If \((f', b')\) and \((f'', b'')\) are normally cobordant highly \( \eta \)-connected degree one normal maps, there is a highly \( \eta \)-connected normal cobordism between \((f', b')\) and \((f'', b'')\). (Again this is true if \( N \) is an \( \eta \)-Poincaré complex over \( B \).) This defines \( \Theta_\eta \).

To eventually complete surgeries in the middle dimension we assume that the map \( p : N \to B \) is \( UV^1 \). Then one has the following (see [PQR; p.3, line 28f1]): Given \( \delta > 0 \) there exists \( \eta > 0 \) such that if \( \Theta_\eta([f', b']) = 0 \), then \((f', b')\) is normally cobordant to a \( \delta \)-homotopy equivalence. If \((f'', b'')\) is normally cobordant to \((f', b')\) by a highly \( \eta \)-connected normal cobordism, then controlled surgery produces a controlled \( h \)-cobordism, giving an \( \varepsilon \)-homotopy by the thin \( h \)-cobordism theorem. This defines an element of \( S_{\varepsilon, \delta}(N, p) \), and shows the semi-exactness of the sequence

\[
S_{\varepsilon, \delta}(N, p) \to [N, G/TOP] \xrightarrow{\Theta_\eta} L_n(B, \mathbb{Z}, \eta, \eta),
\]

i.e. \( \ker \Theta_\eta \) contains the image of \( S_{\varepsilon, \delta}(N, p) \). We note that semi-exactness also holds for \( \eta \)-controlled Poincaré complexes over \( B \).

One cannot expect the sequence (3.4) to be exact, i.e. that the composition map is zero, since passing from topology to algebra one loses control. As it was noted in [PQR; p. 3], \( \varepsilon \) and \( \delta \) are determined by the controlled Hurewicz and Whitehead theorems. Exactness of (3.4) will follow by the Squeezing Lemma (Lemma 3 of [PY]).

The proof of (3.3) will be completed by showing that the assembly map

\[
A : H_n(B, \mathbb{L}) \to L_n(B, \mathbb{Z}, \eta, \eta)
\]

is bijective for sufficiently small \( \eta \). This follows by splitting the controlled quadratic Poincaré complexes (i.e. the elements of \( L_n(B', \mathbb{Z}, \eta, \eta) \)) into small pieces over small simplices of \( B \) (we assume for simplicity that \( B \) is triangulated). If \( \delta \) is given, and if we want a splitting where each piece is \( \delta \)-controlled, we must start the subdivision with a sufficiently small \( \eta \)-controlled quadratic Poincaré complex (see Remark below). This can be done by Lemma 6 of [PQR] (see also [Y; Lemma 2.5]). Since \( A \circ \Theta = \Theta_\eta \), we get (3.3) from (3.4). The stability constant \( \varepsilon_0 \) is determined by the largest \( \eta \) for which \( A \) is bijective.

Remark. Yamasaki has estimated the size of \( \eta \) in the Splitting Lemma. If one performs a splitting so that the two summands are \( \delta \)-controlled, then one needs an \( \eta \)-controlled algebraic quadratic Poincaré complex with \( \eta \) of size \( \delta/\left((an^k + b)\right) \), where \( a, b, k \) depend on \( X \) (\( k \) is conjectured to be 1), and \( n \) is the length of the complex. Of course, squeezing also follows from bijectivity of \( A \) for small \( \eta \), but Lemma 3 of [PY] is somehow a clean statement to apply (see Theorem 3.5 below). We also note that the bijectivity of \( A \) is of course independent of whether \( N \) is a manifold or a Poincaré complex.
Theorem 3.5. Suppose that $N \xrightarrow{p} B$ is a $UV^1$ map. Let $\delta > 0$ be given (sufficiently small, i.e. $\delta < \delta_0$ for some $\delta_0$). Then there is $\eta > 0$ (small with respect to $\delta$), such that if $N$ is an $\eta$–Poincaré complex over $B$, and $(f, b) : M \to N$ is a degree one normal map, then $\Theta(f, b) = 0 \in H_n(B, \mathbb{L})$ if (and only if) $(f, b)$ is normally bordant to a $\delta$–equivalence.

The "only if" part is more delicate and it follows by Lemma 3 of [PY]: So let $f : M^n \to N^n$ be a $\delta$–equivalence which defines a quadratic $\eta_1$–Poincaré complex $C$ in $L_n(B, \mathbb{Z}, \eta_1, \eta_1)$ which is $\eta_1$–cobordant to zero via $[N, G_{\text{TOP}}] \to L_n(B, \mathbb{Z}, \eta_1, \eta_1)$.

Then $C$ is $\kappa\eta_1$–cobordant to an arbitrary small quadratic Poincaré complex (i.e. to a quadratic $\eta$–complex) which is $\kappa\eta_1$–cobordant to zero, with $\eta_1$ sufficiently small (i.e. $\eta$ sufficiently small). In this case we can also assume that $A$ is bijective. This proves the "only if" part.

Theorem 3.5 can also be stated as follows:

Theorem 3.5'. Let $N$ be a sufficiently fine $\eta$–Poincaré complex over a $UV^1$–map $p : N \to B$. Then there exist $\varepsilon > 0$ and $\delta > 0$, both sufficiently small, so that the sequence

$$S_{\varepsilon, \delta}(N, p) \to [N, G_{\text{TOP}}] \to H_n(B, \mathbb{L})$$

is exact. In particular, it holds for generalized manifolds.

Part II. $UV^1$ approximation.

Here we recall Proposition 4.3 and Theorem 4.4. of [BFMW]:

Theorem 3.6. Suppose that $f : (M^n, \partial M) \to B$ is a continuous map from a compact $n$–manifold with boundary such that the homotopy fiber of $f$ is simply connected. If $n \geq 5$ then $f$ is homotopic to a $UV^1$–map. In case that $f \big|_{\partial M}$ is already $UV^1$, the homotopy is relative $\partial M$.

We state the second theorem in the form which we will need. We take it from Theorem 10.1 of [Fe]:

Theorem 3.7. (Ferry) Let $p : N^n \to B$ be a map from a compact $n$–manifold into a polyhedron, $n \geq 5$.

(i) Given $\varepsilon > 0$, there is a $\delta > 0$, such that if $p$ is a $UV^1(\delta)$–map then $p$ is $\varepsilon$–homotopic to a $UV^1$–map.

(ii) Suppose that $p : N \to B$ is a $UV^1$ map. Then for each $\varepsilon > 0$ there is a $\delta > 0$ (depending on $p$ and $\varepsilon$) such that if $f : M \to N$ is a $\delta$–1–connected map (over $B$) from a compact manifold $M$ of dim $M \geq 5$, then $f$ is $\varepsilon$–close over $B$ to a $UV^1$–map $g : M \to N$.

Part III. Controlled gluing.

The following is Proposition 4.6 of [BFMW]:
**Theorem 3.8.** Let $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ be (orientable) manifolds and $p_i : M_i \to B$ $\mathcal{UV}^1$-maps. Then there exist $\varepsilon_0 > 0$ and $T > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $h : \partial M_1 \to \partial M_2$ an (orientation preserving) $\varepsilon$-equivalence, $M_1 \cup_{h} M_2$ is a $T\varepsilon$-Poincaré complex over $B$.

**Part IV. Approximation of retractions.**

The following is Proposition 4.10 of [BFMW]:

**Theorem 3.9.** Let $X$ and $Y$ be finite polyhedra. Suppose that $V$ is a regular neighborhood of $X$ with $\dim V \geq 2 \dim Y + 1$ and $r : V \to X$ is the retraction. If $f : Y \to X$ is an $\varepsilon$-equivalence with respect to $p : X \to B$, then there exists an embedding $i : Y \to V$ and a retraction $s : V \to i(Y)$ with $d(p \circ r, p \circ s) < 2\varepsilon$.

We now begin with the construction. Let $M^n$ be a closed oriented (topological) manifold of dimension $n \geq 6$. Let $\sigma \in H_n(M, \mathbb{L})$ be fixed. Moreover, we assume that $M$ is equipped with a simplicial structure. Then let $M = B \cup_{D} C$ be such that $B$ is a regular neighborhood of the $2$–skeleton, $D = \partial B$ is its boundary and $C$ is the closure of the complement of $B$. So $D = \partial C = B \cap C$ is of dimension $\geq 5$.

By Part II above we can replace $(B, D) \subset M$ and $(C, D) \subset M$, by $\mathcal{UV}$–maps $j : (B, D) \to M$ and $j : (C, D) \to M$, and we can realize $\sigma$ according to $H_n(M, \mathbb{L}) \to S_{\varepsilon, \delta}(D, j)$ by a degree one normal map $F_{\sigma} : V \to D \times I$ with $\partial_0 V = D$, $\partial_1 V = D'$, $F_{\sigma} \big|_{\partial_1 V} = \text{Id}$ and $f_{\sigma} = F_{\sigma} \big|_{\partial_1 V} : D' \to D$ a $\delta$–equivalence over $M$.

We then define $X_0 = B \cup_{F_{\sigma}} -V \cup_{\text{Id}} C$, where $-V$ is the cobordism $V$ turned upside down. We use the map $-F_{\sigma} \cup \text{Id} : -V \cup_{\text{Id}} C \to D \times I \cup C \cong C$ to extend $j$ to a map $p_0 : X_0 \to M$.

The Wall realization $V \to D \times I$ is such that $V$ is a cobordism built from $D$ by adding high–dimensional handles (similarly beginning with $D'$). Therefore $p_0$ is a $\mathcal{UV}$–map: If $(K, L)$ is a simplicial pair with $K$ a $2$–complex, and if there is given a diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\alpha_0} & X_0 \\
\uparrow & & \downarrow p_0 \\
K & \xrightarrow{\alpha} & M
\end{array}
\]

then we first move (by an arbitrary small approximation) $\alpha$ and $\alpha_0$ into $B$ by general position arguments. Then one uses the $\mathcal{UV}$–property of $j : B \to M$. By 3.8, $X_0$ is a $T\delta$–Poincaré complex over $M$. Note that we can choose $\delta$ as small as we want, hence we get an $\eta_0$–Poincaré complex for a prescribed $\eta_0$. This completes the first step.
To continue we define a manifold $M^n_0$ and a degree one normal map $g_0 : M^n_0 \to X_0$ as follows:

$$M_0 = B \cup V \cup -V \cup C \to B \cup D \times I \cup -V \cup C \cong X_0$$

using $F_\sigma \cup \text{Id} : V \cup -V \to D \times I \cup -V$. By construction it has a controlled surgery obstruction $\sigma \in H_n(M, \mathbb{L})$.

Moreover, there is $\overline{\sigma} \in H_n(X_0, \mathbb{L})$ with $p_{0*}(\overline{\sigma}) = \sigma$. This can be seen from the diagram:

$$
\begin{array}{c}
H_n(M_0, \mathbb{L}) \xrightarrow{g_{0*}} H_n(X_0, \mathbb{L}) \xrightarrow{p_{0*}} H_n(M, \mathbb{L}) \\
\cong \ \uparrow \ \cong \\
H^0(M_0, \mathbb{L}) \leftarrow g_{0*} \ H^0(X_0, \mathbb{L}) \leftarrow p_{0*} \ H^0(M, \mathbb{L}).
\end{array}
$$

The vertical isomorphisms are Poincaré dualities. Since $p_0$ is a $UV^1$ map, $\overline{\sigma}$ belongs to the same $\mathbb{Z}$-sector as $\sigma$. We will again denote $\overline{\sigma}$ by $\sigma$.

We construct $p_1 : X_1 \to X_0$ as above: Let $M_0 = B_1 \cup C_1, B_1$ a regular neighborhood of the 2-skeleton (as fine as we want), $C_1$ the closure of the complement and $D_1 = C_1 \cap B_1 = \partial C_1 = \partial B_1, g_0 : D_1 \to X_0$ a $UV^1$ map. Then we realize $\sigma \in H_n(X_0, \mathbb{L}) \to \mathcal{S}_{\varepsilon, \delta_1}(D_1, g_0)$ by $F_{1, \sigma} : V_1 \to D_1 \times I$ with $\partial_0 V_1 = D_1, \partial_1 V_1 = D'_1, F_{1, \sigma}|_{\partial_1 V_1} = \text{Id}$ and $f_{1, \sigma} = F_{1, \sigma}|_{\partial_0 V_1} : D'_1 \to D_1$ a $\delta_1$-equivalence over $X_0$.

We define $p'_1 : X'_1 \to X_0$ as follows:

$$X'_1 = B_1 \cup_{f_{1, \sigma}} -V_1 \cup C_1 \xrightarrow{f'_{1, \sigma}} M_0 \cong B_1 \cup D_1 \times I \cup C_1,$$

using $-F_{1, \sigma} : -V_1 \to D_1 \times I$, and then $p'_1 = g_0 \circ f'_{1} : X'_1 \to M_0 \to X_0$.

We now observe that:

(i) by 3.8 $X'_1$ is a $T_1\delta_1$–Poincaré complex over $X_0$; and

(ii) $p'_1$ is a degree one normal map with controlled surgery obstruction

$$-p_{0*}(\overline{\sigma}) + \sigma = 0 \in H_n(M, \mathbb{L}).$$

Let $\xi_1 > 0$ be given. We now apply Theorem 3.5 to produce a $\xi_1$–homotopy equivalence by surgeries outside the singular set (note that the surgeries which have to be done are in the manifold part of $X'_1$). For this we need a sufficiently small $\eta_0$–Poincaré structure on $X_0$. However, this can be achieved as noted above. This finishes the second step.

We now proceed by induction: What we need for the third step in order to produce $p_2 : X_2 \to X_1$ is the following:

(i) a degree one normal map $g_1 : M_1 \to X_1$ with controlled surgery obstruction $\sigma \in H_n(X_0, \mathbb{L})$; and
(ii) \( \sigma \in H_n(X_1, \mathbb{L}) \) with \( p_1* (\sigma) = \sigma \), being in the \( \mathbb{Z} \)-sector as \( \sigma \in H_n(X_0, \mathbb{L}) \).

One can get \( g_1 : M_1 \rightarrow X_1 \) as follows: Consider \( g'_1 : M'_1 \rightarrow X'_1 \), where

\[
M'_1 = B_1 \cup_{1d} V_1 \cup_{1d} \mathbb{C} \cup_{f_1, \sigma} B_1 \cup_{1d} D_1 \times I \cup_{f_1, \sigma} V_1 \cup_{1d} \mathbb{C} \cong X'_1
\]

is induced by \( F_{1, \sigma} : V_1 \rightarrow D_1 \times I \) and the identity. The map \( g'_1 \) is a degree one normal map. Then one performs the same surgeries on \( g'_1 \) as one has performed on \( p'_1 : X'_1 \rightarrow X_0 \) to obtain \( X'_1 \). This produces the desired \( g_1 \). For (ii) we note that \( p_{1*} \) is a bijective map preserving the \( \mathbb{Z} \)-sectors (since \( p_1 \) is \( UV \).

So we have obtained the sequence of controlled Poincaré spaces \( p_i : X_i \rightarrow X_{i-1} \) and \( p_0 : X_0 \rightarrow M \) with degree one normal maps \( g_i : M_i \rightarrow X_i \) and controlled surgery obstructions \( \sigma \in H_n(X_{i-1}, \mathbb{L}) \). The properties (iv) and (v) of §2 now follow by the thin \( h \)-cobordism theorem and approximation of retraction.

\[ \text{§ 4. Non resolvability, the DDP property} \]
\[ \text{and existence of generalized manifolds} \]

**Part I. Nonresolvability.**

At the beginning of the construction we have \( \sigma \in H_n(M, \mathbb{L}) \), where \( M \) is a closed (oriented) \( n \)-manifold with \( n \geq 6 \). For each \( m \) we constructed degree one normal maps \( g_m : M_m \rightarrow X_m \) over \( p_m : X_m \rightarrow X_{m-1} \), with controlled surgery obstructions \( \sigma_m \in H_n(X_{m-1}, \mathbb{L}) \), \( p_{0*}(\sigma_1) = \sigma \), \( p_{m*}(\sigma_{m+1}) = \sigma_m \), and all \( \sigma_m \) belong to the same \( \mathbb{Z} \)-sector as \( \sigma \). So we will call all of them \( \sigma \).

We consider the normal map \( g_m : M_m \rightarrow X_m \) as a controlled normal map over \( \text{Id} : X_m \rightarrow X_m \), and over \( q_m : X_m \subset W_m \xrightarrow{\rho} X \) (see §2). Since \( \rho |_{\partial W_m} \) is an approximate fibration and \( d(r_i, r_{i-1}) < \xi_i \), \( \sum_{i=m+1}^{\infty} \xi_i < \varepsilon \), for large \( m \), we can assume that \( q_m \) is \( UV \) for large \( m \), so \( (q_m)_* : H_n(X_m, \mathbb{L}) \rightarrow H_n(X, \mathbb{L}) \) maps \( \sigma \) to \( (q_m)_*(\sigma) = \sigma' \), being in the same \( \mathbb{Z} \)-sector as \( \sigma \). The map \( (q_m)_* \) is a bijective, and we denote \( \sigma' \) by \( \sigma \). In other words, we have a surgery problem over \( X \):

\[
M_m \xrightarrow{g_m} X_m \xrightarrow{q_m} X
\]

with controlled surgery obstruction \( \sigma \in H_n(X, \mathbb{L}) \). Our goal is to consider the surgery problem:

\[
M_m \xrightarrow{q_m \circ g_m} X \xrightarrow{\text{Id}} X
\]
over \( \text{Id} : X \to X \), and prove that \( \sigma \in H_n(X, \mathbb{L}) \) is its controlled surgery obstruction. Observe that \( q_m \) is a \( \delta \)-homotopy equivalence over \( \text{Id} : X \to X \) if \( m \) is sufficiently large (for a given \( \delta \)).

Let \( \mathcal{N}(X) \cong [X, G/TOP] \) be the normal cobordism classes of degree one normal maps of \( X \), and let \( HE_\delta(X) \) be the set of \( \delta \)-homotopy equivalences of \( X \) over \( \text{Id} : X \to X \). Our claim will follow from the following:

**Lemma 4.1.** Let \( HE_\delta'(X) \times \mathcal{N}(X) \overset{\mu}{\to} \mathcal{N}(X) \) be the action map, i.e. \( \mu(h, f) = h \circ f \). Then for sufficiently small \( \delta' > 0 \), the diagram

\[
\begin{array}{ccc}
HE_\delta' \times \mathcal{N}(X) & \xrightarrow{\mu} & \mathcal{N}(X) \\
\downarrow \rho & & \downarrow \Theta \\
\mathcal{N}(X) & \overset{\Theta}{\longrightarrow} & H_n(X, \mathbb{L})
\end{array}
\]

commutes.

**Proof.** This follows from (3.5') since \( HE_\delta'(X) \times \mathcal{S}_{\epsilon', \delta'}(X, \text{Id}) \to \mathcal{S}_{\epsilon, \delta}(X, \text{Id}) \) for sufficiently small \( \delta' \) and \( \delta'' \). \( \square \)

We apply this lemma to the map \( HE_\delta(X_m, X) \times \mathcal{N}(X_m) \to \mathcal{N}(X) \), which sends \( (h, g) \) to \( h \circ g \), where \( HE_\delta(X_m, X) \) are the \( \delta \)-homotopy equivalences \( X_m \to X \) over \( \text{Id} : X \):

Let \( \psi_m : X \to X_m \) be a controlled inverse of \( q_m \). Then \( \psi_m \) induces \( \psi_m^* : HE_\epsilon(X_m, X) \to HE_\delta(X) \), where \( \delta \) is some multiple of \( \epsilon \).

One can then write the following commutative diagram (for sufficiently small \( \delta \)).

\[
\begin{array}{ccc}
HE_e(X_m, X) \times H_n(X_m, \mathbb{L}) & \xrightarrow{\text{Id} \times \Theta} & \mathcal{N}(X) \times H_n(X, \mathbb{L}) \\
\downarrow \Theta & & \downarrow \Theta \\
HE_e(X_m, X) \times \mathcal{N}(X_m) & \to & \mathcal{N}(X)
\end{array}
\]

with \( HE_e(X) \times H_n(X_m, \mathbb{L}) \to H_n(X, \mathbb{L}) \) given by \( (h, \tau) \to h \circ (\tau) \).

It follows from this that for large enough \( m \), \( q_m \circ g_m : M_m \to X \) has controlled surgery obstruction \( \sigma \in H_n(X, \mathbb{L}) \). Hence we get non-resolvable generalized manifolds if the \( Z \)-sector of \( \sigma \) is \( \neq 1 \).

**Part II. The DDP Property.**

The construction allows one to get the DDP property for \( X \) (see [BFMW; section 8]). Roughly speaking, this can be seen as follows: The first step in the construction
is to glue a highly connected cobordism \( V \) into a manifold \( M \) of dimension \( n \geq 6 \), in between the regular neighborhood of the \( 2 \)-skeleton.

The result is a space which has the DDP. The other constructions are surgery on middle-dimensional spheres, which also preserves the DDP. But since we have to take the limit of the \( X_m \)'s, one must do it more carefully (see Definition 8.1 in [BFMW]):

**Definition 4.2.** Given \( \varepsilon > 0 \) and \( \delta > 0 \), we say that a space \( Y \) has the \((\varepsilon, \delta)\)-DDP if for each pair of maps \( f, g : D^2 \to Y \) there exist maps \( f, g : D^2 \to Y \) such that \( d(f, f(D^2), g(D^2)) > \delta \), \( d(f, f) < \varepsilon \) and \( d(f, g(D^2)) < \varepsilon \).

**Lemma 4.3.** \( \{X_m\} \) have the \((\varepsilon, \delta)\)-DDP for some \( \varepsilon > \delta > 0 \).

**Proof.** The manifolds \( M_m^n \), \( n \geq 6 \), have the \((\varepsilon, \delta)\)-DDP for all \( \varepsilon \) and \( \delta \). In fact, one can choose a sufficiently fine triangulation, so that any \( f : D^2 \to M \) can be placed by arbitrary small moves into the \( 2 \)-skeleton or into the dual \((n-3)\)-skeleton. Then \( \delta \) is the distance between these skeleta. The remarks above show that the \( X_m \)'s have the \((\varepsilon, \delta)\)-DDP for some \( \varepsilon \) and \( \delta \). \( \Box \)

It can then be shown that \( X = \varprojlim X_i \) has the \((2\varepsilon, \delta/2)\)-DDP (see Proposition 8.4 in [BFMW]).

**Part III. Special cases.**

(i) Let \( M^n \) and \( \sigma \in H_n(M, \mathbb{L}) \) be given as above. The first case which can occur is that \( \sigma \) goes to zero under the assembly map \( A : H_n(M, \mathbb{L}) \to L_n(\pi_1 M) \). Then we can do surgery on the normal maps \( F(\sigma) : V \to D \times I \), \( F_{1, \sigma} : V_1 \to D_1 \times I \) and so on, to replace them by products. In this case the generalized manifold \( X \) is homotopy equivalent to \( M \).

(ii) Suppose that \( A \) is injective (or is an isomorphism). Then \( X \) cannot be homotopy equivalent to any manifold, if the \( Z \)-sector of \( \sigma \) is \( \neq 1 \). Suppose that \( N^n \to X \) were a homotopy equivalence. It determines an element in \([X, G/TOP]\) which must map to \((1, 0) \in H_n(X, \mathbb{L})\), because its surgery obstruction in \( L_n(\pi_1 X) \) is zero and \( A \) is injective. This contradicts our assumption that the index of \( X \neq 1 \). Examples of this type are given by the \( n \)-torus \( M^n = T^n \).

**References**


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