ON LINEAR REALIZATIONS
AND LOCAL SELF-SIMILARITY
OF THE UNIVERSAL
ZARICHNYI MAP

Taras Banakh    Dušan Repovš

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TARAS BANAKH AND DUŠAN REPOVŠ

Abstract. Answering a question M.Zarichnyi we show that the universal Zarichnyi
map $\mu : \mathbb{R}^\infty \to Q^\infty$ is not locally self-similar. Also we characterize linear op-

erators homeomorphic to $\mu$ and on this base give a simple construction of a universal
Zarichnyi map $\mu$.

In this paper we investigate the properties of the universal map $\mu : \mathbb{R}^\infty \to Q^\infty$
constructed by M.Zarichnyi in [Za3] and subsequently studied in [Za4] and [Za5].
Answering a question posed in [Za5] we prove that the map $\mu$ is not locally self-
similar. Also we characterize linear operators homeomorphic to $\mu$ and on this base
give a simple construction of a universal Zarichnyi map $\mu$.

Strongly universal maps

All topological spaces considered in this paper are Tychonov, all compact spaces
are metrizable, and all maps are continuous; $\omega = \{0,1,2,\ldots \}$ stands for the set of
all finite ordinals.

Given a class $\mathcal{C}$ of compacta, by $\mathcal{C}^\infty$ we denote the class of topological spaces $X$
admitting a countable cover $\mathcal{U}$ by subsets of the class $\mathcal{C}$, generating the topology of
$X$ in the sense that a subset $F \subset X$ is closed in $X$ if and only if $F \cap K$ is closed in
$K$ for every $K \in \mathcal{U}$. In our subsequent considerations $\mathcal{C} = \mathcal{K}$ or $\mathcal{K}_{fd}$, where $\mathcal{K}$
($\mathcal{K}_{fd}$) is the class of all (finite-dimensional) metrizable compacta.

Given a space $X$ with a fixed point $*$ let $X^\infty$ denote the set

$$X_f^\omega = \{(x_i)_{i \in \omega} \in X^\omega : x_i = * \text{ for almost all } i\}$$

endowed with the strongest topology inducing the product topology on each space
$X^n = \{(x_i)_{i \in \omega} \in X : x_i = * \text{ for all } i \geq n\}$, $n \in \omega$. If the space $X$ is topologically
homogeneous (like the real line $\mathbb{R}$ or the Hilbert cube $Q = [0,1]^\omega$), then the topology
of the space $X^\infty$ does not depend on the particular choice of a fixed point $* \in X$.

Among the spaces $X^\infty$ the spaces $\mathbb{R}^\infty$ and $Q^\infty$ occupy the special place: they are
universal for the classes $\mathcal{K}_{fd}$ and $\mathcal{K}^\infty$ in the sense that each space from the class $\mathcal{K}^\infty$
(resp. $\mathcal{K}_{fd}$) is homeomorphic to a closed subspace of $Q^\infty$ (resp. $\mathbb{R}^\infty$). Topological
copies of the spaces $\mathbb{R}^\infty$ and $Q^\infty$ very often appear in topological algebra and

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functional analysis, see [Ba$_2$]—[BS$_2$], [Sa$_2$]—[Za$_2$]. In particular, every infinite-dimensional linear topological space $X \in \mathcal{K}_{fd}^\infty$ is homeomorphic to $\mathbb{R}^\infty$ [Ba$_2$] while each locally convex space $Y \in \mathcal{K}^\infty$ with uncountable Hamel basis is homeomorphic to $Q^\infty$ [Ba$_3$]. A topological characterization of the spaces $\mathbb{R}^\infty$ and $Q^\infty$ was given by K. Sakai [Sa$_1$]: Up to a homeomorphism $\mathbb{R}^\infty$ (resp. $Q^\infty$) is a unique strongly $\mathcal{K}_{fd}$-universal (resp. strongly $\mathcal{K}$-universal) space in the class $\mathcal{K}_{fd}^\infty$ (resp. $\mathcal{K}^\infty$).

A topological space $X$ is defined to be strongly $C$-universal if every embedding $f : B \to X$ of a closed subset $B$ of a space $A \subset C$ can be extended to an embedding $\tilde{f} : A \to X$. Replacing the words “embedding” by “map” we obtain the definition of an absolute extensor for the class $C$ (briefly, $\text{AE}(C)$).

In [Za$_3$] the notion of the strong universality was generalized to maps. A map $\pi : X \to Y$ between topological spaces is defined to be strongly $C$-universal if for every embedding $f : B \to X$ of a closed subset $B$ of a space $A \subset C$ and a map $g : A \to Y$ with $\pi \circ f = g|B$ there exists an embedding $\tilde{f} : A \to X$ such that $\tilde{f}|B = f$ and $\pi \circ \tilde{f} = g$. Replacing the words “embedding” by “map” we obtain the definition of a $C$-soft map. Observe that a space $X$ is strongly $C$-universal (resp. is an $\text{AE}(C)$) if and only if the constant map $X \to \{\ast\}$ is strongly $C$-universal (resp. $C$-soft).

The following theorem was proven in [Ba$_1$].

**Uniqueness Theorem.** If $\pi : X \to Y$, $\pi' : X' \to Y$ are strongly $C$-universal maps from spaces $X, X' \in C^\infty$, then there is a homeomorphism $h : X \to X'$ such that $\pi' \circ h = \pi$.

In case of a one-point space $Y$ we obtain Uniqueness Theorem of strongly $C$-universal spaces (see [Sa$_1$]): Any two strongly $C$-universal spaces $X, X' \in C^\infty$ are homeomorphic.

Thus (up to a homeomorphism) there is at most one strongly $\mathcal{K}_{fd}$-universal map from a space $X \in \mathcal{K}_{fd}^\infty$ onto a given space $Y$. For which spaces $Y$ does such a map exist? If $Y \in \mathcal{K}_{fd}^\infty$, then the answer is easy: just consider the projection $\pi : Y \times \mathbb{R}^\infty \to Y$. If $Y \notin \mathcal{K}_{fd}^\infty$ (for example, if $Y = Q^\infty$) the situation is not so obvious. Nonetheless, applying certain non-trivial results of A.Dranishnikov [Dr], M.Zarichnyi has constructed in [Za$_3$] a strongly $\mathcal{K}_{fd}$-universal map $\mu : \mathbb{R}^\infty \to Q^\infty$.

Afterwards, he proved that this map $\mu$ is homeomorphic to a group homomorphism [Za$_3$] and to an affine map between suitable spaces of probability measures [Za$_1$], thus giving an alternative and simpler constructions of the map $\mu$. The Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$ contains any map $f : A \to B$ from a finite-dimensional metrizable compactum $A$ into a metrizable compactum $B$ in the sense that there are two embeddings $e_A : A \to \mathbb{R}^\infty$ and $e_B : B \to Q^\infty$ such that $\mu \circ e_A = e_B \circ f$.

**Characterizing linear operators homeomorphic to the map $\mu$**

We define two maps $\pi : X \to Y$ and $\pi' : X' \to Y'$ to be homeomorphic if $\pi' \circ h = H \circ \pi$ for some homeomorphisms $h : X \to X'$ and $H : Y \to Y'$. In this section we characterize linear operators homeomorphic to the universal Zarichnyi map and classify $\mathcal{K}_{fd}$-invertible linear operators from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ onto a locally convex space $Y \in \mathcal{K}^\infty$. Under a “linear operator” we understand a linear continuous operator between linear topological spaces.

We define a map $\pi : X \to Y$ to be $C$-invertible if for every map $g : A \to Y$ of a space $A \in C$ there is a map $f : A \to X$ such that $\pi \circ f = g$. It is clear that each
strongly $C$-universal or $C$-soft map is $C$-invertible. For linear operators the converse statement is also true. In [Za5, 2.1] M.Zarichnyi proved that a continuous group homomorphism $h : G \to H$ is $K_{fd}$-soft if and only if $h$ is $K_{fd}$-invertible and its kernel $\ker h$ is an AE$(K_{fd})$. Since each linear topological space is an AE$(K_{fd})$ (see, e.g., [Ba2]), we get

**Theorem 1.** A linear operator between linear topological spaces is $K_{fd}$-soft if and only if it is $K_{fd}$-invertible.

Next, we find conditions under which a given linear operator is strongly $K_{fd}$-universal.

**Theorem 2.** A linear operator $T : X \to Y$ from a linear topological space $X \in K_{fd}^\infty$ to a linear topological space $Y$ is strongly $K_{fd}$-universal if and only if the operator $T$ is $K_{fd}$-invertible and has infinite-dimensional kernel.

In the proof we will exploit two lemmas.

**Lemma 1.** If $X \in K_{fd}^\infty$ is an infinite-dimensional linear topological space, then for every compactum $K \subset X$ there is a non-zero point $x \in X$ such that $([-1, 1] \cdot x) \cap K \subset \{0\}$.

**Proof.** Replacing $K$ by $[-1, 1] \cdot K$, if necessary, we may assume that $K = [-1, 1] \cdot K$. Since $K$ is a compact subset of the space $X \in K_{fd}^\infty$, $\dim K < n$ for some $n \in \mathbb{N}$. The linear space $X$, being infinite-dimensional, contains an $n$-dimensional linear space $\mathbb{R}^n$. We claim that there is a point $x$ on the unit sphere $S$ of $\mathbb{R}^n$ such that $([-1, 1] \cdot x) \cap K = \{0\}$. Assuming the converse, for every $x \in S$ we would find a number $n(x) \in \mathbb{N}$ such that $([-1, 1] \cdot x) \cap K \supset [0, \frac{1}{n(x)}] \cdot x$. It can be shown that for every $n \in \mathbb{N}$ the set $S_n = \{x \in S : n(x) \leq n\}$ is closed in $S$. Since $S = \bigcup_{n \in \mathbb{N}} S_n$, the Baire Theorem guarantees that $S_n$ has non-empty interior in $S$ for some $n$. Then $\dim S_n = n - 1$. Since $K \supset [0, 1/n] \cdot S_n$ and $n > \dim K > \dim ([0, 1/n] \cdot S_n)$, we get a contradiction. □

**Lemma 2.** If $T : X \to Y$ is a linear operator with infinite-dimensional kernel from a linear topological space $X \in K_{fd}^\infty$, then for every compact subset $C \subset X$ there exists an embedding $c : C \times [0, 1] \to X$ such that $c(c, 0) = c$ and $T \circ c(c, t) = T(c)$ for all $c \in C$ and $t \in [0, 1]$.

**Proof.** Let $L = \ker T$ and $K = L \cap (C - C)$. By Lemma 1, there exists a non-zero point $x_0 \in L$ such that $([-1, 1] \cdot x_0) \cap K \subset \{0\}$. Define the map $c : C \times [0, 1] \to X$ letting $c(c, t) = c + tx_0$ for $(c, t) \in C \times [0, 1]$. It is clear that $c(c, 0) = c$ and $T \circ c(c, t) = T(c)$ for every $c \in C$ and $t \in [0, 1]$. To show that the map $c$ is injective, fix two points $(c, t), (c', t') \in C \times [0, 1]$ with $c(c, t) = c(c', t')$. Then $c - c' = (t' - t)x_0$ and $T(c - c') = 0$ which implies $c - c' \in K$. Since $K \supset c - c' = (t' - t)x_0 \in [-1, 1] \cdot x_0$ and $K \cap \{(c, t) : (c, t) \in K, \} \subset \{0\}$, we get $c - c' = 0$ and $t' - t = 0$, i.e., $(c, t) = (c', t')$. □

**Proof of Theorem 2.** The “only if” part of Theorem 2 is trivial. To prove the “if” part, assume that $T : X \to Y$ is an $K_{fd}$-invertible operator, $X \in K_{fd}^\infty$, and $\dim \ker T = \infty$. To prove the strong $K_{fd}$-universality of the map $T$, fix an embedding $f : B \to X$ of a closed subset $B$ of a space $A \in K_{fd}$ and a map $q : A \to Y$ such that $T \circ f = g|B$. By Theorem 1, the operator $T$ is $K_{fd}$-soft and hence there is a map $\tilde{f} : A \to X$ such that $\tilde{f}|B = f$ and $T \circ \tilde{f} = g$.

Denote by $A/B$ the quotient space and let $q : A \to A/B$ be the quotient map. It is clear that the space $A/B$ is finite-dimensional and thus admits an embedding
$i : A/B \to [0,1]^n$ for some $n \in \mathbb{N}$ such that $e(\{B\}) = 0^n \in [0,1]^n$. Applying Lemma 2 several times, construct an embedding $\varepsilon : \tilde{f}(A) \times [0,1]^n \to X$ such that $e(x,0^n) = x$ and $T \circ e(x,t) = T(x)$ for any $x \in \tilde{f}(A)$ and $t \in [0,1]^n$. It is easy to verify that the map $\tilde{f} : A \to X$ defined by $\tilde{f}(a) = e(\tilde{f}(a),i \circ q(a))$ for $a \in A$ is an embedding satisfying the conditions $\tilde{f}|B = f$ and $T \circ \tilde{f} = g$. □

We apply Theorem 2 to prove the following theorem characterizing linear operators homeomorphic to the universal Zarichnyi map $\mu$.

**Theorem 3.** A linear operator $T : X \to Y$ between linear topological spaces is homeomorphic to the strongly $K_{fd}$-universal Zarichnyi map $\mu$ if and only if $X \in K^\infty_{fd}$, $Y$ is homeomorphic to $Q^\infty$, and the operator $T$ is $K_{fd}$-invertible.

**Proof.** This theorem will follow from Theorem 2 and Uniqueness Theorem for strongly $K_{fd}$-universal map as soon as we prove that each $K_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in K^\infty_{fd}$ onto a linear topological space $Y$ containing a Hilbert cube has infinite-dimensional kernel. Assume to the contrary that $\text{Ker} T$ is finite-dimensional. By Theorem 1, the operator $T$, being $K_{fd}$-invertible, is $K_{fd}$-soft. Fix a copy $Q \subset Y$ of the Hilbert cube in $Y$ and an open surjective map $g : A \to Q$ of a finite-dimensional compactum $A$ onto $Q$ (such a map exists according to [Dr]). Since the operator $T$ is $K_{fd}$-invertible, there is a map $f : A \to X$ such that $T \circ f = g$. Let $B \subset \text{Ker} T$ be any compact neighborhood of the origin in the finite-dimensional linear space $\text{Ker} T$. Next, consider the compact set $K = f(A) + B \subset X$. Since $X \in K^\infty_{fd}$, $\dim K < n$ for some $n \in \mathbb{N}$. Let $e : I^n \to Q \subset Y$ be any embedding of the $n$-dimensional cube $I^n$ into $Q$. By the $K_{fd}$-softness of the map $T$, there is a map $i : I^n \to X$ such that $T \circ i = e$ and $i(0^n) \in f(A)$. It is clear that $i$ is an embedding.

We claim that $K$ is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$. Assuming that it is not true, we would find a sequence $(x_n)_{n=1}^\infty \in i(I^n) \setminus K$ tending to $x_0$. Then the sequence $(T(x_n))_{n=1}^\infty$ converges to $T(x_0)$. Let $a_0 \in A$ be any point with $f(a_0) = x_0$. Since the map $g : A \to Q$ is open and $g(a_0) = T \circ f(a_0) = T(x_0) = \lim_{n \to \infty} T(x_n)$, there exists a sequence $(a_n)_{n=1}^\infty \subset A$ tending to $a_0$ such that $g(a_n) = T(x_n)$ for each $n$. Then the sequence $(f(a_n))_{n=1}^\infty$ converges to $f(a_0) = x_0$ and has the property: $T \circ f(a_n) = g(a_n) = T(x_n)$ for every $n$. Hence $x_n - f(a_n) \in \text{Ker} T$ for every $n$. Since $\lim_{n \to \infty} x_n = x_0 = \lim_{n \to \infty} f(a_n)$, we get $\lim_{n \to \infty} (x_n - f(a_n)) = 0$ and thus $x_n - f(a_n) \in B$ for some $n$. Then $x_n \in B + f(a_n) \subset B + f(A) = K$, a contradiction with the choice of the sequence $(x_n)_{n=1}^\infty$. Thus $K$ is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$ what is not possible since $\dim K < n = \dim(U)$ for any neighborhood $U \subset i(I^n)$ of $i(0^n)$. This contradiction shows that the kernel of $T$ is infinite-dimensional. □

We remind that a topological space $Y$ is called a $k$-space if a subset $F \subset Y$ is closed in $Y$ if and only if for every compact subset $K \subset Y$ the intersection $F \cap K$ is closed in $K$.

**Theorem 4.** A $K_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in K^\infty_{fd}$ onto a locally convex $k$-space $Y$ is homeomorphic either to the projection $pr : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for some $n, m \in \omega \cup \{\infty\}$ or to the strongly $K_{fd}$-universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$. The latter case occurs if and only if the space $Y$ has uncountable Hamel basis.
ON THE UNIVERSAL ZARICHNYI MAP

Proof. First, we show that $Y \in \mathcal{K}^\infty$. Since $X \in \mathcal{K}^\infty_{fd}$, the space $X$ contains a countable collection $\{X_n : n \in \omega\}$ of compact subsets, fundamental in the sense that every compact subset of $X$ lies in some $X_n$. By [En1, 3.1.22], for every $n$ the image $f(X_n)$ of the compact set $X_n$ is a metrizable compactum. The operator $T$, being $\mathcal{K}_{fd}$-invertible, is surjective. Then $Y = \bigcup_{n \in \omega} f(X_n)$. We claim that the collection $\{f(X_n) : n \in \omega\}$ generates the topology of $Y$, i.e., $Y \in \mathcal{K}^\infty$.

Assuming the converse, we would find a non-closed subset $F \subset Y$ such that $F \cap f(X_n)$ is closed in $f(X_n)$ for every $n$. Since $Y$ is a $k$-space, there is a compactum $K \subset Y$ such that $F \cap K$ is not closed in $K$. The compactum $K = \bigcup_{n \in \omega} K \cap f(X_n)$, being the countable union of metrizable compacta, is metrizable. Consequently, there is a sequence $(y_n)_{n=1}^\infty \subset F \cap K$ converging to a point $y_0 \in K \setminus F$. Since the map $T$ is $\mathcal{K}_{fd}$-invertible, there is a sequence $(x_n)_{n=1}^\infty \subset X$ converging to a point $x_0 \in X$ such that $T(x_n) = y_n$ for all $n \in \omega$. The subset $\{x_n : n \in \omega\} \subset X$, being compact, lies in the compactum $X_m$ for some $m$. Then $\{y_n : n > 0\} \subset f(X_m) \cap F$. Since the intersection $f(X_m) \cap F$ is closed in $f(X_m)$, we get $y_0 = \lim_{n \to \infty} y_n \in f(X_m) \cap F$, a contradiction with $y_0 \notin F$.

Thus the locally convex space $Y$ belongs to the class $\mathcal{K}^\infty$. By [Ba3], $Y$ is homeomorphic either to $Q^\infty$ or to $\mathbb{R}^n$ for some $n \in \omega \cup \{\infty\}$. Moreover, the last case occurs if and only if the algebraic dimension of $Y$ is at most countable. If $Y$ is homeomorphic to $Q^\infty$, then by Theorem 2, the operator $T$ is homeomorphic to the universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$.

If $Y$ is homeomorphic to $\mathbb{R}^n$ for some $n \in \omega \cup \{\infty\}$, then the algebraic dimension of $Y$ is at most countable and $Y$ carries the strongest linear topology, see [Ba2]. In this case there is a linear continuous operator $S : Y \to X$ such that $T \circ S = id$ and the map $h : X \to Y \times \text{Ker} T$ defined by $h(x) = (T(x), x - S \circ T(x))$ for $x \in X$ is a linear homeomorphism (with inverse $h^{-1}(y, l) = S(y) + l$) such that $pr \circ h = T$, where $pr : Y \times \text{Ker} T \to Y$ is the projection. Hence the operator $T$ is homeomorphic to the projection $pr : Y \times \text{Ker} T \to Y$. Since $\text{Ker} T$ is a linear topological space from the class $\mathcal{K}^\infty_{fd}$ we can apply Corollary 1 (or [Ba2]) to conclude that $\text{Ker} T$ is homeomorphic to $\mathbb{R}^m$ for some $m \in \omega \cup \{\infty\}$. Therefore $T$ is homeomorphic to the projection $pr : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. \[\Box\]

Next, we show that linear operators classifying by Theorem 3 do exist.

**Theorem 5.** For every linear topological space $Y \in \mathcal{K}^\infty$ there is a $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}^\infty_{fd}$.

Proof. Since $Y \in \mathcal{K}^\infty$, the space $Y$ possesses a countable family $\{F_n : n \in \omega\}$ of compact subsets such that every compact subset of $Y$ lies in some $F_n$. For every $n \in \omega$ denote by $\mathcal{K}_n$ the subclass of $\mathcal{K}_{fd}$ consisting of all at most $n$-dimensional compacta. Using the Dranishnikov Theorem [Dr], for every $n$ we can find an $\mathcal{K}_n$-invertible map $f_n : K_n \to F_n$ of a finite-dimensional metrizable compactum $K_n$ onto $F_n$. Now consider the discrete sum $K = \sqcup_{n \in \omega} K_n$ and the map $f = \sqcup_{n \in \omega} f_n : K \to Y$. It is clear that $K \in \mathcal{K}^\infty_{fd}$ and the map $f$ is $\mathcal{K}_{fd}$-invertible. Let $L(K) \supset K$ be the free linear topological space over $K$ and $T : L(K) \to Y$ unique linear operator extending the $\mathcal{K}_{fd}$-invertible map $f$. It is clear that the operator $T$ is $\mathcal{K}_{fd}$-invertible. By [Ba2], $L(K) \in \mathcal{K}^\infty_{fd}$. \[\Box\]

Note that Theorems 4 and 5 allow us to give an alternative construction of the universal map of Zarichnyi.
Question. Let \( h : G \to H \) be a \( \mathcal{K}_{f,d} \)-invertible continuous homomorphism between topological groups. Is \( h \) strongly \( \mathcal{K}_{f,d} \)-universal if its kernel \( h^{-1}(1) \) is strongly \( \mathcal{K}_{f,d} \)-universal?

The universal Zarichnyi map is not self-similar

We define a space \( X \) to be **locally self-similar** if every point of \( X \) has a basis of neighborhoods homeomorphic to \( X \). It is well-known that the spaces \( \mathbb{R}^\infty \) and \( Q^\infty \) (like many other model spaces of infinite-dimensional topology) are locally self-similar.

In [Za5] M. Zarichnyi extended the notion of the local self-similarity onto maps and asked if the map \( \mu : \mathbb{R}^\infty \to Q^\infty \) is locally self-similar. He defined a map \( \pi : X \to Y \) to be **locally self-similar** if for every point \( x \in X \) and every neighborhood \( U \subset X \) of \( x \) there is a neighborhood \( V \subset U \) of \( x \) such that the map \( \pi|V : V \to \pi(V) \) is homeomorphic to \( \pi \). Observe that for locally self-similar spaces \( X, Y \) the projection \( \text{pr} : X \times Y \to Y \) is locally self-similar.

The following result shows that the strongly \( \mathcal{K}_{f,d} \)-universal Zarichnyi map is close to being locally self-similar.

**Theorem 6.** If \( T : X \to Y \) is a linear operator homeomorphic to the strongly \( \mathcal{K}_{f,d} \)-universal Zarichnyi map \( \mu : \mathbb{R}^\infty \to Q^\infty \), then for every nonempty open convex subset \( U \subset X \) the map \( T|U : U \to T(U) \) is homeomorphic to \( \mu \).

**Proof.** Fix any non-empty open convex subset \( U \subset X \). Since \( X \) is homeomorphic to \( \mathbb{R}^\infty \), \( U \) is homeomorphic to \( \mathbb{R}^\infty \) too. Next, \( T(U) \) is homeomorphic to \( \mathbb{R}^\infty \), being open contractible subspace of \( Y \), the topological copy of \( Q^\infty \). By Uniqueness Theorem, to show that the map \( T|U : U \to T(U) \) is homeomorphic to \( \mu \) it suffices to verify that \( T|U \) is a strongly \( \mathcal{K}_{f,d} \)-universal map.

First we show that the map \( T|U : U \to T(U) \) is \( \mathcal{K}_{f,d} \)-invertible. Fix any map \( g : A \to T(U) \) from a compactum \( A \subset \mathcal{K}_{f,d} \). Since the map \( T \) is \( \mathcal{K}_{f,d} \)-invertible, there is a map \( f : A \to X \) such that \( T \circ f = g \). For every point \( x \in K \) find a point \( a_x \in \text{Ker} T \) such that \( f(x) + a_x \in U \). Next, let \( U(x) = f^{-1}(U - a_x) \). Using the compactness of \( A \), find a finite subcover \( \{U(x) : x \in A\} \) and let \( \{\lambda_i : A \to [0, 1]\}_{i=1}^{n} \) be a partition of unity such that \( \lambda_i^{-1}(0, 1) \subset U(x_i) \) for every \( i \leq n \). Consider the map \( \alpha : A \to \text{Ker} T \) defined by \( \alpha(x) = \sum_{i=1}^{n} \lambda_i(x)a_x \), for \( x \in K \). It can be shown that the map \( h = f + \alpha : A \to X \) has the properties: \( T \circ h = T \circ f = g \) and \( h(A) \subset U \). Therefore, the map \( T|U \) is \( \mathcal{K}_{f,d} \)-invertible.

To show that it is strongly \( \mathcal{K}_{f,d} \)-universal, fix an embedding \( f : B \to U \) of a closed subset \( B \) of a space \( A \subset \mathcal{K}_{f,d} \) and a map \( g : A \to T(U) \) such that \( T \circ f = g \). Since the map \( T|U : U \to T(U) \) is \( \mathcal{K}_{f,d} \)-invertible, there is a map \( h_1 : K \to U \) such that \( T \circ h_1 = g \). Next, using the strong \( \mathcal{K}_{f,d} \)-universality of the operator \( T \), find a map \( h_2 : A \to X \) such that \( h_2|B = f \) and \( T \circ h_2 = g \). Let \( \lambda : A \to [0, 1] \) be a continuous map such that \( \lambda(B) = \{0\} \) and \( \lambda(A \setminus W) = \{1\} \), where \( W = h^{-1}(U) \). Next, consider the map \( h : A \to X \) defined by \( h(a) = \lambda(a)h_1(a) + (1 - \lambda(a))h_2(a) \) for \( a \in A \). It is easy to see that \( T \circ h = g \) and \( h(A) \subset U \).

Let \( q : A \to A/B \) be the quotient map. Since the quotient space \( A/B \) is finite-dimensional, there is an embedding \( i : A/B \to [0, 1]^n \) for some \( n \in \omega \) such that \( i(B) \cap [0, 1]^n = \emptyset \). The space \( h(A) \) belongs to the class \( \mathcal{K}_{f,d} \), being a compact subset of the space \( X \subset \mathcal{K}^\infty_{f,d} \). Then by the strong \( \mathcal{K}_{f,d} \)-universality of the operator \( T \), there is
an embedding $e : h(A) \times [0, 1]^n \to X$ such that $e(x, 0^n) = x$ and $T \circ e(x, t) = T(x)$ for every $x \in h(A)$ and $t \in [0, 1]^n$. Since $h(A) \subset U$, there exists $\varepsilon > 0$ such that $e(h(A) \times [0, \varepsilon]^n) \subset U$. Finally consider the map $\tilde{f} : A \to X$ defined by $\tilde{f}(a) = e(h(a), \varepsilon \cdot o q(a))$ for $a \in A$. It can be easily shown that $\tilde{f}|U = f$, $\tilde{f}(A) \subset U$ and $T \circ \tilde{f} = g$. Therefore, the map $T|U$ is strongly $K_{fd}$-universal. The space $U$, being an open subset of the space $X \in K_{fd}^\infty$, belongs to the class $K_{fd}^\infty$. If the space $T(U)$ is homeomorphic to $Y$, then by Uniqueness Theorem the maps $T|U$ and $T$ are homeomorphic. □

Thus the local self-similarity of the Zarichnyi map $\mu$ would be proven if we would find a linear operator between locally convex spaces, homeomorphic to $\mu$. Unfortunately, no such an operator exists. This is so because each locally convex space $X \in K_{fd}^\infty$ has at most countable Hamel basis, see [Ba2]. Consequently, each linear image of $X$ also has at most countable Hamel basis and thus cannot be homeomorphic to $Q^\infty$. But this is not a unique reason why we cannot find a linear operator between locally convex spaces, homeomorphic to the universal Zarichnyi map $\mu$.

**Theorem 6.** Each open $K_{fd}$-invertible map $f : \mathbb{R}^\infty \to Q^\infty$ is not self-similar.

Let us define a map $\pi : X \to Y$ to be locally $K_{fd}$-invertible if for every point $x \in X$ and every neighborhood $U \subset X$ of $x$ there is a neighborhood $V \subset U$ of $x$ such that the map $\pi|V : V \to \pi(V)$ is $K_{fd}$-invertible. It is clear that each locally self-similar $K_{fd}$-invertible map is locally $K_{fd}$-invertible.

We define a space $X$ to be almost finite-dimensional if there is $n \in \mathbb{N}$ such that $\dim F \leq n$ for every finite-dimensional closed subset $F$ of $X$. Let us note that there exist infinite-dimensional almost finite-dimensional compact spaces, see [En2, 5.2.23].

Theorem 6 will be derived from

**Lemma 3.** Each compactum $K$ admitting a surjective locally $K_{fd}$-invertible map $f : X \to K$ from a space $X \in K_{fd}^\infty$, is almost finite-dimensional.

To prove this lemma we need

**Lemma 4.** For every compact space $K$ and a closed subset $A \subset K \times Q$ with $\dim A < \dim K$ there is a closed subset $F \subset K \times Q$ such that $F \cap A = \emptyset$ but $F \cap s(K) \neq \emptyset$ for every section $s : K \to K \times Q$ of the projection $pr : K \times Q \to K$.

**Proof.** Find any $n \in \omega$ with $\dim A \leq n < \dim K$. By Hurewicz-Wallman Theorem [En2, 1.9.3], there exists a map $f : L \to S^n$ from a closed subset $L$ of $K$ into the $n$-dimensional sphere which has no continuous extension $\tilde{f} : K \to S^n$.

Since $S^n$ is an ANR, we can extend $f$ to a continuous map $\tilde{f} : \tilde{O}(L) \to S^n$ defined on the closure of an open neighborhood $O(L)$ of $L$ in $K$. Let $B = L \times Q$ and $U = O(L) \times Q$. Since $\dim A \leq n$, we can apply Hurewicz-Wallman Theorem again to find a continuous map $p : A \to S^n$ such that $p|A \cap \tilde{U} = \tilde{f} \circ pr|A \cap \tilde{U}$. Next, since $S^n$ is an ANR, there is a continuous map $\tilde{p} : V \to S^n$ defined on an open neighborhood $V$ of the closed set $A \cap \tilde{U}$ in $K \times Q$ such that $\tilde{p}|A = p$ and $\tilde{p}|\tilde{U} = \tilde{f} \circ pr|\tilde{U}$. Consider the open set $O(A) = O(B) \cup (V \setminus B)$, where $O(B) = \{x \in U : d(\tilde{p}(x), \tilde{f} \circ pr(x)) < 1\}$ and $d$ stands for the standard Euclidean metric of $\mathbb{R}^{n+1} \supset S^n$.

We claim that the closed set $F = (K \times Q) \setminus O(A)$ misses $A$ but meets the image $s(K)$ of every any section $s : K \to K \times Q$ of the projection $pr$. Assuming the
converse, we would find a section $s : K \to K \times Q$ of the projection $\text{pr}$ such that $s(Q) \cap F = \emptyset$. Then $s(K) \subset O(A)$ and we can consider the map $g = \bar{p} \circ s : K \to S^n$. Observe that $s(L) \subset B \cap O(A) = O(B)$ and hence $d(g(x), f(x)) = (d(\bar{p} \circ s(x)), \bar{f} \circ \text{pr} \circ s(x)) < 1$ for all $x \in L$ which yields that the maps $g|L$ and $f$ are homotopic. Since $g|L$ has the extension $g : K \to S^n$, we may apply the Borsuk Extension Theorem [En, 1.9.7] to conclude that the map $f$ has a continuous extension $\bar{f} : K \to S^n$, which contradicts to the choice of $f$. □

Proof of Lemma 3. Suppose a compactum $K$ admits an open surjective locally $\mathcal{K}_{f,d}$-invertible map $\pi : X \to K$ of a space $X \in \mathcal{K}_{f,d}$. Assume that the compactum $K$ is not almost finite-dimensional. The compactum $K$, being a countable union of metrizable compacta, is metrizable. By the compactness of $K$ there is a point $y \in K$ having no almost infinite dimensional neighborhood in $K$. Fix a countable base $\{U_n : n \in \omega\}$ of neighborhoods of $y$ in $K$ and for every $n \in \omega$ find a finite dimensional compactum $K_n \subset U_n$ with $\dim K_n > n$.

Fix any point $x \in X$ with $\pi(x) = y$ and let $\{X_n : n \in \omega\}$ be an increasing collection of finite-dimensional compact subsets of $X$ generating its topology. Without loss of generality, $x \in X_0$ and $\dim X_n \leq n$ for every $n \in \omega$. The space $X$, having a countable network of the topology, admits an injective continuous map $i : X \to Q$ into the Hilbert cube. Now consider the injective map $e = (\pi, i) : X \to K \times Q$ defined by $e(x) = (f(x), i(x))$ for $x \in X$. For every $n \in \omega$ let $A_n = i(X_n) \cap (K_n \times Q)$. Since $\dim A_n \leq \dim X_n \leq n < \dim K_n$, we can apply Lemma 4 to find a closed subset $F_n \subset K_n \times Q$ such that $A_n \cap F_n = \emptyset$ but $F_n \cap s(K_n) \neq \emptyset$ for every section $s : K_n \to K_n \times Q$ of the projection $\text{pr} : K_n \times Q \to K_n$.

Consider the set $F = \bigcup_{k \in \omega} e^{-1}(F_k)$. Since each set $F_k$ is closed in $K \times Q$ and $X_n \cap F = X_n \cap (\bigcup_{k=0}^{n-1} e^{-1}(F_k))$ for each $n \in \omega$, we get that $F$ is a closed subset of $X$. Next, since $X_0 \cap F = \emptyset$, the set $U = X \setminus F$ is an open neighborhood of the point $x$ in $X$.

Let us show that for every open neighborhood $V \subset U$ of $x$ the map $\pi|V : V \to \pi(V)$ is not $\mathcal{K}_{f,d}$-invertible. Indeed, since the map $\pi$ is open, $\pi(V)$ is an open neighborhood of the point $\pi(x) = y$ and thus $f(V) \supset K_n$ for some $n \in \omega$. Assuming that the map $\pi|V$ is $\mathcal{K}_{f,d}$-invertible we would find a map $g : K_n \to V \subset U$ such that $\pi \circ g = \text{id}$. Then the map $s = e \circ g : K_n \to K \times Q$ has the properties: $\text{pr} \circ s = \text{id}$ and $s(K_n) \cap F_n = \emptyset$, a contradiction with the choice of the set $F_n$. □

Proof of Theorem 6. Assume that $f : \mathbb{R}^\omega \to Q^\omega$ is an open $\mathcal{K}_{f,d}$-invertible locally self-similar map. Then $f$ is locally $\mathcal{K}_{f,d}$-invertible. Let $K \subset Q^\omega$ be a topological copy of the Hilbert cube and $X = f^{-1}(K)$. Then $X \in \mathcal{K}_{f,d}$ and $f|K : X \to K$ is an open surjective locally $\mathcal{K}_{f,d}$-invertible map, a contradiction with Lemma 3. □
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Department of Mathematics and Mechanics, Lviv University, Universytetska 1, Lviv, 79000, Ukraine
E-mail address: tbanakh@franko.lviv.ua

Institute of Mathematics, Physics and Mechanics, Jadranska 19, Ljubljana, Slovenia 1001
E-mail address: dusan.repovs@uni-lj.si