ON TOPOLOGICAL HELLY THEOREM

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Abstract. The main result of this paper is the following theorem, related to the missing link in the proof of the topological version of the classical result of Helly:

Let \( \{X_i\}_{i=0}^{2} \) be any family of simply connected compact subsets of \( \mathbb{R}^2 \) such that for every \( i, j \in \{0, 1, 2\} \) the intersections \( X_i \cap X_j \) are path connected and \( \bigcap_{i=0}^{2} X_i \) is nonempty. Then for every two points in the intersection \( \bigcap_{i=0}^{2} X_i \) there exists a cell-like compactum connecting these two points, in particular the intersection \( \bigcap_{i=0}^{2} X_i \) is a connected set.

1. Introduction

A topological space \( X \) is said to be simply connected if it is path connected and has a trivial fundamental group, \( \pi_1(X) = 1 \). It is well-known that for every subspace \( X \subset \mathbb{R}^2 \) of the plane, \( \pi_1(X) = 1 \) if and only if for every Jordan curve \( \mathcal{J} \subset X \) and every point \( y \in \mathbb{R}^2 \setminus \mathcal{J} \) from the bounded component of \( \mathbb{R}^2 \setminus \mathcal{J} \), \( y \) lies in \( X \) (see e.g. [Ku2; Ch. 10, §61, II, Theorem 5] or [Wh; p. 107, Proposition 2.51]) or equivalently, no Jordan curve \( \mathcal{J} \subset X \) is a retract of \( X \).

Throughout this paper all singular and Čech (co)homology groups will be assumed to have the integer coefficients \( \mathbb{Z} \). A topological space \( X \) is called a singular cell if all its singular homology groups are trivial, \( H_*(X) = H_*(pt) \). Next, \( X \) is said to be acyclic if all its Čech cohomology groups are trivial, \( H^*(X) = H^*(pt) \). A planar compactum is acyclic if and only if it is cell-like (see, e.g. [Da]). A space \( X \) is said to be cell-like connected if for every two points \( a \) and \( b \) there exists cell-like continuum \( C \) in \( X \) such that \( a, b \in C \).

If a subspace \( X \subset \mathbb{R}^2 \) of the plane is not simply connected then, as it was mentioned above, \( X \) contains a Jordan curve \( \mathcal{J} \subset X \) which is a retract of \( X \) and therefore the group \( H_1(X) \) cannot be trivial. If a space \( X \) is simply connected then by the Hurewicz Theorem (see, e.g. [Sp; Theorem VII.5.5]), all homotopy groups of \( X \) are naturally isomorphic to the corresponding singular homology groups of \( X \). However, all planar spaces are aspherical (see, [Za, CCZ]). Therefore a subspace \( X \) of the plane \( \mathbb{R}^2 \) is a singular cell if and only if \( X \) is simply connected.

On the other hand, there exist simply connected spaces which are not acyclic (e.g. the Warsaw circle, see [Na; p. 5]). The following classical result is due to Helly (see e.g. [DGK], [Ec], [He]):

\[
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\]

\textit{Key words and phrases.} Simply connected planar sets, planar absolute retracts, Helly-type theorems, planar continua, acyclicity, asphericity, singular cells, cell-like connectedness.
Theorem 1.1 (Topological Helly Theorem). Let $\mathcal{K} = \{K_i\}_{i=0}^m$, $m \geq n$, be any finite family of closed subsets of the $n$-dimensional Euclidean space $\mathbb{R}^n$ such that the intersection of every $k$ members of $\mathcal{K}$ is a singular cell, for every $k \leq n$, and is nonempty, for $k = n + 1$. Then the intersection $\bigcap_{i=0}^m K_i$ is a singular cell.

All known proofs of Theorem 1.1. are inductive and the initial step (i.e. when $m = n = 2$) is based on the following assertion:

Any family $\{X_i\}_{i=0}^2$ of three simply connected compact subsets of the plane $\mathbb{R}^2$ has a simply connected intersection provided that the intersection $X_i \cap X_j$, $i, j \in \{0, 1, 2\}$ of any two of its members is path connected and the intersection $\bigcap_{i=0}^2 X_i$ of all three members is nonempty.

Apparently, for several years nobody questioned the validity of Assertion (*). However, Bogatyti [Bo; p. 399] has recently pointed out that no complete proof of (*) can be found in the existing literature.

Any intersection $\bigcap_{\lambda \in \Lambda} X_\lambda$ of simply connected subsets $X_\lambda \subset \mathbb{R}^2$, $\lambda \in \Lambda$, has a trivial fundamental group, with respect to any of its points. Indeed, consider any Jordan curve $J \subset \bigcap_{\lambda \in \Lambda} X_\lambda$. Since by hypothesis, every element $X_\lambda$, of the family is simply connected, the bounded region of $\mathbb{R}^2$ determined by $J$ is a subset of $X_\lambda$, therefore it is a subset of the intersection $\bigcap_{\lambda \in \Lambda} X_\lambda$ for every $\lambda \in \Lambda$. Consequently, the fundamental group of the intersection is trivial, $\pi_1(\bigcap_{\lambda \in \Lambda} X_\lambda) = 1$. Hence, in order to prove Assertion (*) it is necessary to verify that the intersection $\bigcap_{i=0}^2 X_i$ of all three sets is path connected. In the present paper we provide the first step towards filling this gap – by establishing the cell-like connectedness of $\bigcap_{i=0}^2 X_i$:

Theorem 1.2. A family $\{X_i\}_{i=0}^2$ of three simply connected compact subsets of $\mathbb{R}^2$ has a cell-like connected intersection $\bigcap_{i=0}^2 X_i$ provided that the intersection $X_i \cap X_j$, $i, j \in \{0, 1, 2\}$, of any two of its members is path connected and the intersection $\bigcap_{i=0}^2 X_i$ of all three is nonempty.

The corresponding result for acyclic spaces can be found in [Ch], [De], [Ei], and [Ku1]. The requirement from Assertion (*) that the intersection $X_i \cap X_j$ of any two of the sets be path connected cannot be weakened to just connectedness – as the following result from [KRZ] demonstrates:

Theorem 1.3. There exist three simply connected compact subsets of $\mathbb{R}^2$ such that intersection of any of two of these sets is connected and the intersection of all three of them is a disconnected two-point set.

2. Preliminaries

Lemma 2.1. Let $A$ and $B$ be disjoint subcontinua of a compactum $X$. Then there exists a continuum $C \subset X$ such that $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$ if and only if the inclusion-induced homomorphism $\varphi : \check{H}^0(X) \to \check{H}^0(A \cup B)$ of the Čech cohomology groups is not an epimorphism.

Proof. ($\Rightarrow$) Suppose that there exists a continuum $C \subset X$ connecting $A$ and $B$, i.e. $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. Then obviously, $\check{H}^0(A \cup B \cup C) \cong \mathbb{Z}$. Since $A \cap B = \emptyset$ it follows that $\check{H}^0(A \cup B) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the composition of the inclusion-induced homomorphisms $\check{H}^0(X) \to \check{H}^0(A \cup B \cup C) \to \check{H}^0(A \cup B)$ cannot be an epimorphism.
(⇐) Conversely, let $U$ be a clopen (i.e. open and closed) subset of $X$ which contains $A$. Such a set always exists, take for example $X$. Let $C$ be the intersection of all such sets (i.e. $C$ is the quasi-component of the set $A$). Note that the quasi-component of any compact space is always a continuum (see e.g. [Ku2; Ch. 5, §47. II, Theorem 2]).

Suppose that $B \cap C = \emptyset$. Then there exists a clopen set $U \subset X$ which contains $A$ and does not intersect $B$. Recall that zero-dimensional Čech cohomology $H^0(Y)$ is always naturally isomorphic to the group of locally constant functions from $Y$ into the group of integers $\mathbb{Z}$ with the discrete topology. Now, since $A$ and $B$ are connected and $U$ is clopen in $X$, any locally constant function on $A \cup B$ in this case can be extended over $A \cup B \cup U$ and hence over $X$. Therefore $\varphi$ must be an epimorphism. Contradiction. □

**Lemma 2.2.** Let $C$ and $D$ be acyclic subcontinua of the plane $\mathbb{R}^2$. Then each component of connectedness of the intersection $C \cap D$ is an acyclic continuum.

**Proof.** Consider the cohomology Mayer-Vietoris exact sequence:

$$\ldots \to \tilde{H}^1(C) \oplus \tilde{H}^1(D) \to \tilde{H}^1(C \cap D) \to \tilde{H}^2(C \cup D) \to \ldots$$

Since $C$ and $D$ are acyclic spaces and $C \cup D$ is a planar set we have that $\tilde{H}^1(C) \cong \tilde{H}^1(D) \cong \tilde{H}^2(C \cup D) \cong 0$. It follows that $\tilde{H}^1(C \cap D) \cong 0$. Again by the Mayer-Vietoris exact sequence it follows that for every quasi-component $A$ of $C \cap D$ the first cohomology vanishes, $\tilde{H}^1(A) \cong 0$. Since in compact spaces every quasi-component is component and for any planar set $M$ the higher Čech cohomologies are trivial, $\tilde{H}^n(M) = 0, n \geq 2$, it follows that every component of $C \cap D$ is an acyclic space. □

Let $\Delta_n, n \in \mathbb{N}$, be the standard $n$-dimensional simplex $[e_0e_1 \cdots e_n]$ with vertices $e_0, e_1, \ldots, e_n$. Let $I^{n+1}$ be the $(n+1)$-dimensional prism $\Delta_n \times [0,1]$. Let $I_{[i_0 \ldots i_n]}$, $0 \leq k \leq n$, be its $(k+1)$-dimensional face $[e_{i_0}e_{i_1} \cdots e_{i_k}] \times [0,1]$, generated by the vertices $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$. Denote by $A = \Delta_n \times \{1\}$ and $B = \Delta_n \times \{0\}$ the top and the bottom faces of the prism, respectively. Let $J_i = A \cup B \cup I_i$, where $I_i = I_{[i_1 \ldots i_n]}$ is the $n$-dimensional face generated by all vertices $e_0, e_1, \ldots, e_n$, except the vertex $e_i$.

The following result is of its own interest and its special case for $n = 2$ will play the key role in the proof of our Theorem 1.3:

**Proposition 2.3.** Suppose that the prism $I^{n+1}$ is covered by a family $\{F_i\}_{i=0}^n$ of closed sets and that for every $i$, the face $I_i$ is contained in $F_i$. Then there exists a continuum $C \subset \bigcap_{i=0}^n F_i$ such that $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$.

3. **Proof of Proposition 2.3 : Special Case**

First, suppose that $J_i \subset F_i$. By Lemma 2.1 it suffices to prove that the inclusion-induced homomorphism $H^0(\bigcap_{i=0}^n F_i) \to H^0(A \cup B)$ is not an epimorphism. From the Mayer-Vietoris exact sequence for the pair $(\bigcap_{i=0}^{n-k-1} F_i, \bigcup_{j=n-k}^{n} F_j)$ and the equalities:

$$\bigcap_{i=0}^n F_i = (\bigcap_{i=0}^{n-1} F_i) \cap F_n,$$

$$(\bigcap_{i=0}^{n-k} F_i) \cup (\bigcup_{j=n-k+1}^{n} F_j) = (\bigcap_{i=0}^{n-k-1} F_i \cup (\bigcup_{j=n-k+1}^{n} F_j)) \cap (F_{n-k} \cup (\bigcup_{j=n-k+1}^{n} F_j)),$$
and
\[ (\cap_{i=0}^{n-k-1} F_i \cup (\cup_{j=n-k+1}^{n} F_j)) \cup (F_{n-k} \cup (\cup_{j=n-k+1}^{n} F_j)) = (\cap_{i=0}^{n-k-1} F_i) \cup (\cup_{j=n-k}^{n} F_j), \]
we get for \( k = 0 \) the natural boundary homomorphism:
\[ \check{H}^0(\bigcap_{i=0}^{n} F_i) \to \check{H}^1((\bigcap_{i=0}^{n} F_i) \cup F_n), \]
whereas for every \( 1 \leq k \leq n \) we obtain the homomorphisms:
\[ \check{H}^k \left( \left( \bigcap_{i=0}^{n-k} F_i \right) \cup \left( \bigcup_{j=n-k+1}^{n} F_j \right) \right) \to \check{H}^{k+1} \left( \left( \bigcap_{i=0}^{n-k} F_i \right) \cup \left( \bigcup_{j=n-k}^{n} F_j \right) \right). \]
The composition of these homomorphisms for \( k = 0, 1, \ldots, (n - 1) \) yields the following homomorphism:
\[ \check{H}^0(\bigcap_{i=0}^{n} F_i) \to \check{H}^n(\bigcup_{j=0}^{n} F_j). \]
By the Mayer-Vietoris exact sequence for the pair \((\cap_{i=0}^{n-k} F_i, \cap_{j=n-k}^{n} F_j)\) we obtain for \( k = 0 \) the following natural epimorphism:
\[ \check{H}^0(\bigcap_{i=0}^{n} J_i) \to \check{H}^1((\bigcap_{i=0}^{n} J_i) \cup J_n) \to 0 \]
and for every \( k \in \{1, \ldots, n\} \) the following epimorphisms:
\[ \check{H}^k \left( \left( \bigcap_{i=0}^{n-k} J_i \right) \cap \left( \bigcup_{j=n-k+1}^{n} J_j \right) \right) \to \check{H}^{k+1} \left( \left( \bigcap_{i=0}^{n-k} J_i \right) \cap \left( \bigcup_{j=n-k}^{n} J_j \right) \right) \to 0 \]
since the spaces \( \bigcap_{i=0}^{n-k} J_i \) and \( \bigcup_{j=n-k}^{n} J_j \) are contractible for every \( k = 0, 1, \ldots, (n - 1) \).
Since \( \bigcap_{i=0}^{n} J_i = A \cup B \), the composition of these homomorphisms for \( k = 0, 1, \ldots, n \) gives an epimorphism \( \delta : \check{H}^0(A \cup B) \to \check{H}^n(\bigcup_{j=0}^{n} J_j) \). So we obtain the following commutative diagram:
\[
\begin{array}{ccc}
\check{H}^0(\bigcap_{i=0}^{n} F_i) & \longrightarrow & \check{H}^n(\bigcup_{j=0}^{n} F_j) \\
\downarrow_{\varphi^0} & & \downarrow_{\varphi^n} \\
\check{H}^0(A \cup B) & \stackrel{\delta}{\longrightarrow} & \check{H}^n(\bigcup_{j=0}^{n} J_j) \longrightarrow 0.
\end{array}
\]
Since \( \bigcup_{j=0}^{n} F_j = P^{n+1} \) and \( P^{n+1} \) is a contractible space, the group \( \check{H}^n(\bigcup_{j=0}^{n} F_j) \) is trivial and so the homomorphism \( \varphi^n \) must also be trivial. However, the epimorphism \( \delta \) is not trivial since \( \bigcup_{j=0}^{n} J_j = \partial(P^{n+1}) \) and \( \check{H}^n(\bigcup_{j=0}^{n} J_j) \cong \mathbb{Z} \). Therefore \( \varphi^0 \) cannot be an epimorphism. Hence by Lemma 2.1 there must exist a continuum \( C \subset \bigcap_{i=0}^{n} F_i \) which connects \( A \) and \( B \).
4. Proof of Proposition 2.3: General Case

Suppose now that \( I_i \subset F_i \). Let \( G_i = F_i \cup A \cup B \). As we have already proved in Chapter 3, there exists a continuum \( C \subset \cap_{i=0}^n G_i \) which connects \( A \) and \( B \). Let \( C_0 = C \cap \cap_{i=0}^n F_i \) and let \( C_x \) be the component of the point \( x \) in the space \( C_0 \). Let \( M \) be a clopen set in \( C_0 \) containing \( C_x \). Then \( M \) intersects either \( A \) or \( B \). Indeed, if \( M \cap A = \emptyset \) and \( M \cap B = \emptyset \) then for some open in \( C \) set \( U \) we would get \( M = C_0 \cap U = C_0 \cap (U \setminus (A \cup B)) = U \setminus (A \cup B) \) and \( M \) would be clopen in \( C \). However, this is impossible since \( C \) is a continuum.

It follows that \( C_x \) must intersect \( A \cup B \). Consider now the union \( \cup_{x \in C_0 \cap A} C_x \). This space is closed in \( C_0 \). Indeed, consider any limit point \( x_0 \) of the set \( \cup_{x \in C_0 \cap A} C_x \).

Let \( M \) be a clopen set in \( C_0 \) containing \( x_0 \). Since sets \( C_x \) are connected it follows that either \( C_x \subset M \) or \( C_x \cap M = \emptyset \). Since \( x_0 \) is the limit point there exists \( x \) such that \( C_x \subset M \). It follows that \( M \cap A \neq \emptyset \). Thus \( C_{x_0} \cap A \neq \emptyset \) and \( \cup_{x \in C_0 \cap A} C_x \) is a closed in \( C_0 \), hence a compact space.

Similarly, \( \cup_{x \in C_0 \cap B} C_x \) is a compact space. It follows that

\[
C \subset \left( A \cup \left( \bigcup_{x \in C_0 \cap A} C_x \right) \right) \cup \left( B \cup \left( \bigcup_{x \in C_0 \cap B} C_x \right) \right)
\]

and since \( C \) is connected

\[
\left( A \cup \left( \bigcup_{x \in C_0 \cap A} C_x \right) \right) \cap \left( B \cup \left( \bigcup_{x \in C_0 \cap B} C_x \right) \right) \neq \emptyset.
\]

Therefore for some \( x \), \( C_x \cap A \neq \emptyset \) and \( C_x \cap B \neq \emptyset \). So there again exists a continuum \( C_x \subset \cap_{i=0}^n F_i \) which connects \( A \) and \( B \). □

5. Proof of Theorem 1.2

Consider any two points \( a \) and \( b \) of the intersection \( \cap_{i=0}^2 X_i \). Since \( X_i \cap X_{i+1} \) is path connected (indices are considered mod 3) there exists an arc \( \gamma_i \) in \( X_i \cap X_{i+1} \), connecting \( a \) and \( b \). The union \( \gamma_i \cup \gamma_{i+1} \subset X_i \cap X_{i+1} \) is a Peano continuum. Let \( R_{i+1} \) be the unbounded complementary domain of \( \gamma_i \cup \gamma_{i+1} \) in the plane \( \mathbb{R}^2 \). Let \( C_{i+1} \) be the union of \( \gamma_i \cup \gamma_{i+1} \) with all bounded complementary domains. The boundaries of \( R_i \) and \( C_i \) are the same. It follows by the characterization theorem for planar continua [Wh, p. 113] that \( C_i \) is a simply connected Peano continua, for every \( i \).

By the Borsuk theorem [Br, Theorem 13.1, chapter V] we can therefore conclude that all \( C_i \) are \( AR^2 \)'s. Since \( X_i \) is simply connected, it follows that \( C_i \subset X_i \), for every \( i \).

We shall associate to points \( a \) and \( b \) of the intersection \( \cap_{i=0}^2 C_i \), the mapping \( f : I^3 \to \bigcup_{i=0}^2 C_i \) of the prism \( I^3 \) in the following manner. Let \( f^0 \) map the faces \( A \) and \( B \) (defined in Chapter 2) to points \( a \) and \( b \), respectively. Let \( f^1 \) be a mapping \( f^1 : A \cup B \cup (\bigcup_{i=0}^2 I_i) \to \bigcup_{i=0}^2 X_i \), which maps \( I_i \), \( i \in \{0,1,2\} \), bijectively on the corresponding \( \gamma_i \).

Since the sets \( C_i \) are simply connected there exists a mapping \( f^2 : \partial (I^3) \to \bigcup_{i=0}^2 C_i \) which is an extension of \( f^1 \). Now, all planar subsets are known to be aspherical ([Za], see also [CCZ]), so there exists an extension \( f : I^3 \to \bigcup_{i=0}^2 C_i \) of the mapping \( f^2 \) such that \( J_i \subset f^{-1}(C_i) \).
By Proposition 2.3 there exists a continuum $C \subset \bigcap_{i=0}^{2} f^{-1}(C_i)$ which connects $A$ and $B$. Then $f(C) \subset \bigcap_{i=0}^{2} C_i$ and $f(C)$ is a continuum. By Lemma 2.2 the component of connectedness of $\bigcap_{i=0}^{2} C_i$ containing $f(C)$ is acyclic and therefore a cell-like continuum connecting $a$ and $b$ in $\bigcap_{i=0}^{2} X_i$. Since $a$, $b$ were arbitrary points of the intersections $\bigcap_{i=0}^{2} X_i$ it follows that this intersection is a cell-like connected set.  □.

6. Epilogue

We remark that a special case of Assertion (*), namely for Peano continua, has recently been verified by Bogatyi [Bo]:

**Theorem 6.1.** Any finite family of simply connected Peano continua in $\mathbb{R}^2$ has a nonempty simply connected intersection, provided that intersection of any two of its members is connected and the intersection of any three of its members is nonempty.

Bogatyi's proof of Theorem 6.1 is based on the following technical lemma [Bo; p. 395]:

**Lemma 6.2.** Suppose that the square $[0,1] \times [0,1]$ is a union of two closed sets $B_0$ and $B_1$ such that $\{i\} \times [0,1] \subset B_i$, $i \in \{0,1\}$. Then there exists a continuum $C \subset B_0 \cap B_1$ such that $C \cap ([0,1] \times \{i\}) \neq \emptyset$, $i \in \{0,1\}$.

We wish to point out that Lemma 6.2 follows from our Proposition 2.3 (for $n = 1$). We shall conclude the paper by the following conjecture, a positive answer to which would prove Assertion (*).

**Conjecture 6.3.** Every component of the intersection of any finite family of planar AR's is an AR.

Note that there exist two topological disks $X_1$ and $X_2$ in $\mathbb{R}^3$ such that the intersection $X_1 \cap X_2$ is homeomorphic to the Topologist's Sine Curve $T$ and hence is not an AR. Indeed, let $X_1$ be the square $[0,1] \times [0,1] \times \{0\} \subset \mathbb{R}^3$. The set $T$ can be considered as a subspace of $X_1$. Let $X_2$ be the square $X_1$ slightly deformed in such a way that only the points which do not belong to $T$ are moved to the points with the same first and second coordinates and with the positive third coordinate. Obviously, such a deformation always exists and the intersection $X_1 \cap X_2$ is clearly homeomorphic to $T$, as asserted.

There also exist two Peano continua $Y_1$ and $Y_2$ in $\mathbb{R}^2$ such that the intersection $Y_1 \cap Y_2$ is homeomorphic to the Topologist's Sine Curve $T$ and hence is not an AR. Let's demonstrate this: define the following subsets of the plane:

$$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \text{ and } y = 0 \text{ or } y = 1/n, \ n \in \mathbb{N}\},$$

$$B_{n,m} = \{(m/2^n,y) \in \mathbb{R}^2 \mid y \in [0,1/2^{n-1}], \ 0 < m < 2^n, \ n \in \mathbb{N}\},$$

$$C_{n,m} = \{(m/3^n,y) \in \mathbb{R}^2 \mid y \in [0,1/2^{n-1}], \ 0 < m < 3^n, \ n \in \mathbb{N}\},$$

and

$$D_n = \{(((-1)^n + 1)/2,y) \in \mathbb{R}^2 \mid y \in [1/2^n,1/2^{n-1}], \ n \in \mathbb{N}\}.$$

Define the planar Peano continua $Y_1$ and $Y_2$ as follows:

$$Y_1 = A \cup (\bigcup_{n=1}^{\infty} D_n) \cup (\bigcup_{n=1}^{\infty} (\bigcup_{0<m<2^n} B_{n,m}))$$
and

\[ Y_2 = A \cup \left( \bigcup_{n=1}^{\infty} D_n \right) \cup \left( \bigcup_{0<m<2^n} C_{n,m} \right). \]

Obviously, \( Y_1 \cap Y_2 = A \cup \left( \bigcup_{n=1}^{\infty} D_n \right) \cong T \), so our assertion follows.

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