ON UNIQUENESS OF CARTESIAN PRODUCTS OF SURFACES WITH BOUNDARY

J. Malešič D. Repovš W. Rosicki
A. Zastrow

ISSN 1318-4865

January 27, 2003

Ljubljana, January 27, 2003
On uniqueness of Cartesian products of surfaces with boundary

J. Malesić, D. Repovš, W. Rosicki and A. Zastrow
January 29, 2003

Abstract. It is known that if one of the factors of a decomposition of a manifold into Cartesian product is an interval then the decomposition is not unique. We prove that the decomposition of a 4-manifold (possibly with boundary) into 2-dimensional factors is unique, provided that the factors are not products of 1-manifolds.¹

1 Introduction

In 1945 Borsuk [B2] showed that any connected compact n-dimensional manifold without boundary has at most one decomposition into a Cartesian product of factors of dimension ≤ 2. If we consider Cartesian products of higher-dimensional manifolds then such uniqueness property does not hold (see Theorem 11.5 in [C-R] and [K-R]). Even if we consider the classical Ulam problem [U] of uniqueness of Cartesian squares, one can find counterexamples for 3-manifolds (cf. [K-S]).

The uniqueness of the decomposition into Cartesian products fails if the factors are 2-manifolds with boundary. A torus with a hole and a disk with two holes are not homeomorphic, however, their Cartesian products with the interval $I=[0,1]$ are homeomorphic.

Similarly, the product of a Möbius band with a hole and the interval $I$ is homeomorphic to the product of a Klein bottle with a hole and the interval $I$. All 2-manifolds in the examples above can be constructed by identifying two pairs of disjoint arcs in the boundary of a disk. After multiplication by the interval $I$, the order of identified arcs on the boundaries of disks becomes inessential. If 3-manifold or more general 3-polyhedron has two different decompositions into Cartesian product then one of the factors in these decompositions must be an interval (see [R2]).

Key words: Cartesian product, Kiemeth formula, prime 2-manifold, splitting theorem, sufficiently large 3-manifold, essential torus, surface.
The uniqueness property holds for Cartesian squares (cf. [F]) and Cartesian powers (cf. [R3]) of 2-manifolds with boundary. The uniqueness (up to permutation of factors) of a Cartesian product of circles and intervals is obvious. We have the uniqueness of decomposition into a finite Cartesian product of 1-polyhedra (cf. [B1]) and 1-dimensional locally connected continua (cf. [C]). A Cartesian product of 1-polyhedra does not have another decomposition into a Cartesian product of polyhedra of dimension \( \leq 2 \) (cf. [R4]). Before we begin to consider uniqueness of Cartesian products of connected 2-manifolds with boundary we need some preliminaries.

**Definition 1.1** Let \( X \) be a compact connected 2-manifold with non-empty boundary. We associate to \( X \) the following number:

\[
s(X) = \text{rank} H_1(X) - \text{rank} H_1(\partial X) + 1.
\]

**Lemma 1.1** Let \( X, Y, X', \) and \( Y' \) be any compact connected 2-manifolds with non-empty boundary and suppose that the Cartesian products \( X \times Y \) and \( X' \times Y' \) are homeomorphic. Then

\[
s(X) s(Y) = s(X') s(Y').
\]

**Proof.** We use an argument similar to the one in [R3], Theorem 2.1. We consider the map

\[
i_* : H_2(X \times Y) \to H_2(X \times Y, \partial(X \times Y)),
\]

which is induced by the inclusion of the pair \( (X \times Y, \emptyset) \). The image of this map is generated by all products \( \zeta_i \otimes \zeta_j \) such that \( \zeta_i \in H_1(X) \) and \( \zeta_j \in H_1(Y) \), such that \( j_k \zeta_{i_k} \neq 0 \), for \( k = 1, 2 \), where

\[
j_{1*} : H_1(X) \to H_1(X, \partial X) \quad \text{and} \quad j_{2*} : H_1(Y) \to H_1(Y, \partial Y)
\]

are given by inclusions. The number \( s(X) \) is equal to \( \text{rank im } j_{1*} \) and the number \( s(Y) \) is equal to \( \text{rank im } j_{2*} \). So \( s(X) s(Y) \) is equal to \( \text{rank im } i_* \). Hence if \( X \times Y \) and \( X' \times Y' \) are homeomorphic it follows that \( s(X) s(Y) = s(X') s(Y') \).

**Lemma 1.2** Let \( X, Y, X', \) and \( Y' \) be any compact connected 2-manifolds with non-empty boundary and suppose that the Cartesian products \( X \times Y \) and \( X' \times Y' \) are homeomorphic. Then with respect to the order of the factors we have:

(i) \( H_1(X) = H_1(X') \) and \( H_1(Y) = H_1(Y') \)

(ii) \( H_1(X, \partial X) = H_1(X', \partial X') \) and \( H_1(Y, \partial Y) = H_1(Y', \partial Y') \).

**Proof.** Let \( H_1(X) = \mathbb{Z}^x, H_1(Y) = \mathbb{Z}^y, H_1(X') = \mathbb{Z}^{x'} \) and \( H_1(Y') = \mathbb{Z}' \). By the Künneth formula we conclude that:
\[ Z^{x+y} \cong H_2(X \times Y) \cong H_2(X' \times Y') \cong Z^{x'y'} \text{ and} \\
Z^{x+y} \cong H_1(X \times Y) \cong H_1(X' \times Y') \cong Z^{x+y}. \]

Hence, \( x = x' \) and \( y = y' \) or \( x = y' \) and \( y = x' \). We can assume that the first case holds. This completes the proof of (i).

If \( X \) is orientable then \( H_1(X, \partial X) = Z^x \). If it is not then \( H_1(X, \partial X) = Z^{x-1} \oplus Z_2 \). Similarly, for \( Y, X' \), and \( Y' \). By the relative K"unneth formula,

\[ H_2(X \times Y, \partial(X \times Y)) = Z^{xy} \oplus Z_2, \]

where \( o_1 = 1 \) if \( X \) is nonorientable and \( o_1 = 0 \) if \( X \) is orientable, and \( o_2 = 1 \) if \( Y \) is nonorientable and \( o_2 = 0 \) if \( Y \) is orientable. Similarly for \( X' \) and \( Y' \).

Hence \( xy - o_2x - o_1y + o_2o_2 = xy - 0_2x - 0_1y + 0_10_2 \). So, if \( x > 1 \) and \( y > 1 \) then \( H_1(X, \partial X) = H_1(X', \partial X') \) and \( H_1(Y, \partial Y) = H_1(Y', \partial Y') \).

If \( x = 0 \) then \( X \) and \( X' \) are homeomorphic to the disk. Therefore \( Y \) and \( Y' \) are both orientable or both nonorientable, and their relative first homology groups are the same.

If \( x = 1 \) then \( X \) can be the annulus \( A = S^1 \times I \) or the Möbius band \( M \). Similarly for \( X' \).

If \( X \) is an annulus then \( H_2(X \times Y, \partial(X \times Y)) = Z \oplus H_1(Y, \partial Y) = H_1(Y, \partial Y) \). If \( X' \) is a Möbius band then \( H_2(X' \times Y', \partial(X' \times Y')) = Z_2 \oplus H_1(Y', \partial Y') \). These groups can be isomorphic only if \( H_1(Y, \partial Y) = Z_2 \) and if \( H_1(Y', \partial Y') \) is equal to \( Z \) or \( Z_2 \). The spaces \( A \times M \) and \( M \times M \) are not homeomorphic by Lemma 1.1; by definition \( s(A) = 0 \), and \( s(M) = 1 \), so \( s(A)s(M) \neq s(M)s(M) \).

We start the consideration of the Cartesian products of connected 2-manifolds with boundary by presenting the case where one of the factors is not prime. In this paper a prime manifold is a manifold which is not a nontrivial Cartesian product. There exist three non-prime surfaces: \( I \times I, I \times S^1 \), and \( S^1 \times S^1 \). We have the following

**Proposition 1.1** Let \( X \) and \( Y \) be any compact 2-manifolds, possibly with boundary, and suppose that the Cartesian products \( X \times Y \) and \( X' \times Y' \) are homeomorphic. If \( X \) is prime and \( Y \) is a product of two 1-manifolds, then \( X' \) is also a prime 2-manifold and \( Y' \) is a product of two 1-manifolds (up to a permutation of \( X' \) and \( Y' \)). In both cases, \( Y \) and \( Y' \) are homeomorphic. Furthermore, if \( X \) and \( X' \) are not homeomorphic, then \( Y \) and \( Y' \) are homeomorphic either to \( I^2 \) or to \( S^1 \times S^1 \).

**Proof.** By Kosiński’s theorem [K], all 2-dimensional Cartesian factors of a polyhedron are polyhedra, so \( X' \) and \( Y' \) are 2-manifolds, possibly with boundary. If \( \partial X = \emptyset \) and \( Y' = S^1 \times S^1 \) then we have the uniqueness by a classical result of Borsuk [B2].

If \( \partial X = \emptyset \) and \( Y' = I \times S^1 \), then the factors \( X', Y' \), say \( X' \) has an empty boundary, because \( H_0(X' \times Y'; Z_2) = H_0(X' \times Y'; Z_2) \neq 0 \). Since
\[ \partial(X \times Y) = X \times \partial Y = X' \times \partial Y' = \partial(X' \times Y'), \] the surfaces \(X\) and \(X'\) are homeomorphic. Hence, comparing the homology groups we obtain that \(Y'\) is an annulus, also.

Now, let \(\partial X = \emptyset\) and \(Y = \mathbb{T}^2\). If \(X\) is non-orientable then \(0 = H_2(X) = H_2(X \times Y) = H_2(X' \times Y')\), so one of the factors \(X', Y'\) is a disk. The second factor is homeomorphic to \(X\). If \(X\) is orientable, \(\partial X' \neq \emptyset\) and \(\partial Y' \neq \emptyset\) then \(Z = H_2(X) = H_1(Y') \oplus H_1(Y')\). Therefore \(X'\) and \(Y'\) are homeomorphic to \(S^1 \times I\) and \(X\). If \(\partial X' = \emptyset\) then the boundaries \(\partial(X \times Y)\) and \(\partial(X' \times Y')\) are homeomorphic, so \(X\) and \(X'\) are homeomorphic and \(Y'\) is a disk.

If \(\partial X \neq \emptyset\) and \(Y = S^1 \times S^1\), then \(Y' = S^1 \times S^1\) because \(\partial(X \times Y)\) is a disjoint union of the sets homeomorphic to \(S^1 \times S^1 \times S^1\). Hence \(X\) and \(X'\) are homeomorphic by a special case of Theorem 2 [R4]. If \(Y\) is homeomorphic to a disk or to an annulus and \(\partial X \neq \emptyset\), then by Lemma 1.2, \(Y'\) is also homeomorphic to a disk or to an annulus.

## 2 The Main Result

The following is the main result of our paper:

**Theorem 2.1** Any connected 4-dimensional manifold, possibly with boundary, has at most one decomposition into Cartesian products of prime 2-manifolds, possibly with boundary.

The techniques which were used in a similar lemma in [R1] are not strong enough for our purpose. We shall use the Splitting Theorem in the proof of our theorem above (see [J-S], [Jo]) for investigation of the boundaries of the manifolds \(X \times Y\) and \(X' \times Y'\). So we use this theorem in the case when \(\partial M\) is empty.

In [J-S], [Jo] manifolds are orientable, so we must also assume that the manifold \(M\) is orientable. We denote by \(\sigma_W(M)\) the 3-manifold obtained by splitting \(M\) along \(W\). Similarly we define the 2-manifold \(\sigma_W(\partial M)\), which can be naturally identified with a submanifold of the boundary of \(\sigma_W(M)\).

**Theorem 2.2** (Splitting Theorem [J-S] p.157) Let \(M\) be any compact, orientable, sufficiently-large, irreducible and boundary-irreducible 3-manifold. Then there exists a two-sided, incompressible 2-manifold, \(W\) properly embedded in \(M\), unique up to ambient isotopy, having the following three properties:

(a) The components of \(W\) are annuli and tori, and none of them is boundary-parallel in \(M\);

(b) Each component of \((\sigma_W(M), \sigma_W(\partial M))\) is either a Seifert pair or a simple pair; and

(c) \(W\) is minimal with respect to inclusion among all two-sided 2-manifolds in \(M\) having properties (a) and (b).
Proof of Theorem 2.1. If both surfaces $X$ and $Y$ are without boundary, the uniqueness holds by Borsuk’s theorem [B2].

If $\partial X = \emptyset$ and $\partial Y \neq \emptyset$ then $\partial(X \times Y) = X \times \partial Y$. Since $Y \neq I^2$, like in the proof of Proposition 1.1, one of the factors $X'$, $Y'$, say $X'$ has an empty boundary, because $H_0(X \times Y; \mathbb{Z}_2) = H_0(X' \times Y'; \mathbb{Z}_2) \neq 0$ and $\partial Y' \neq \emptyset$. So, $\partial(X' \times Y') = X' \times \partial Y'$. Therefore $X$ and $X'$ are homeomorphic and the numbers of the components of the boundaries $\partial Y$ and $\partial Y'$ are the same. Looking at the homology and relative homology groups we obtain that the surfaces $Y$ and $Y'$ are also homeomorphic.

Now we consider the case when $\partial X$ and $\partial Y$ are nonempty. Again by Lemma 1.2, the first Betti numbers of $X$ and $X'$ are the same and the first Betti numbers of $Y$ and $Y'$, are also the same. The coincidence of the first relative homology groups implies that the orientability of $X$ and $Y$ agree with the orientability of $X'$ and $Y'$, respectively. We consider three cases.

In the first case, $X$ and $Y$ are orientable, $M = \partial(X \times Y)$, $W = \partial X \times \partial Y$. Since by assumption, $X$ and $Y$ are not homeomorphic to $I^2$ or $S^1 \times I$, the manifolds $M$ and $W$ satisfy the hypotheses of the Splitting Theorem. Since the boundary of $M$ is empty, the manifold $W$ is a disjoint union of tori.

For somebody who is familiar with 3-manifolds the irreducibility of $M$ is a simple exercise, but for the reader’s convenience we outline a proof. If $S$ is a 2-sphere contained in $M$ we can assume that it is in a general position with $W$, so the intersection $S \cap W$ is a disjoint union of closed curves. Some of them bound inessential disks in $S$. Such a disk lies in one of components of $\sigma_W(M)$. The boundaries of the components are incompressible (II.2.4 [J-S]), so the boundary of the disk bound a disk in $W$. The components of $\sigma_W(M)$ are irreducible (II.2.3 [J-S]), so the union of our two disk bounds a ball. Via this ball we isotope parts of $S$ into the adjacent component of of $\sigma_W(M)$ eliminating one closed curve of $S \cap W$. We repeat this operation as many times as $S$ lies in one component and it bounds a ball.

We will show that $W$ is minimal. Assume that $V = W \setminus (S_1 \times S_2)$ where $S_1 \times S_2$ is a component of $W$ also gives a splitting in the sense of Theorem 2.2. According to $V$, we have $U = (X \times S_2) \cup (S_1 \times Y)$ as a component of $\sigma_V(M)$. It must be either a Seifert pair or a simple pair. The set $U$ is not a simple pair because the incompressible torus $S_1 \times S_2$ is not boundary-parallel in $U$ (see [J-S], p.154).

The fundamental group of $U$ is infinite, so by Corollary 8.3 in [G] or VI.11.a in [J], the manifold $U$ is a Seifert manifold if and only if its fundamental group has a normal cyclic infinite subgroup. Let an element $a$ of $\pi_1(U)$ be a generator of this subgroup. By Seifert-van Kampen theorem $\pi_1(U)$ is a sum with amalgamation of the groups $\pi_1(X \times S_2)$ and $\pi_1(S_1 \times Y)$. The natural projections map the element $a$ onto elements of the centers of $\pi_1(X \times S_2)$ and $\pi_1(S_1 \times Y)$. So, if $\pi_1(X)$ and $\pi_1(Y)$ have more than one generator, it is impossible.

The same holds for $X'$ and $Y'$, where $M' = \partial(X' \times Y')$, $W' = \partial X' \times \partial Y'$. The components of $\sigma_W(M)$ are homeomorphic to spaces $X \times S^1$ and $S^1 \times Y$. Because the manifolds $M$ and $M'$ are homeomorphic and $W$ is unique up to
ambient isotopy, the components of $\sigma_W(M)$ and the components of $\sigma_W(M')$ are homeomorphic. The components of $\sigma_W(M')$ are homeomorphic to spaces $X' \times S^1$ and $S^1 \times Y'$, so the manifolds $X$ and $Y$ are homeomorphic to $X'$ and $Y'$.

In the second case only one manifold is orientable. Let $X$ be non-orientable and $Y$ be orientable. We consider the oriented double covers $\hat{X}$ and $\hat{X}'$ of $X$ and $X'$. The manifolds $\hat{X} \times Y$, and $\hat{X}' \times Y'$ are orientable double covers of the homeomorphic manifolds $X \times Y$ and $X' \times Y'$, so our manifolds are homeomorphic.

If $X$ is the Möbius band, then $X'$ is also nonorientable and $H_1(X) = H_1(X') = Z$, by Lemma 1.2, so $X'$ is the Möbius band, too.

If $X$ is not the Möbius band, then as before, we have homeomorphy either according to $X \approx X'$ and $Y \approx Y'$ or according to $X \approx Y'$ and $Y \approx X'$ by the Splitting theorem. In the first case $X$ and $X'$ are also homeomorphic. In the second case if $H_1(X) = Z^2$ then $H_1(Y) = Z^{2n-1}$. Putting $s(X') = s(X) + a$, $s(Y') = s(Y) + b$, $s(\hat{X}) = 2(s(X) - 1)$ and $s(\hat{X}') = 2(s(X') - 1)$ to the equations

$$s(X)s(Y) = s(X')s(Y')$$
$$s(X)s(Y) = s(X')s(Y')$$

we obtain $s(Y) = s(Y')$, so $Y$ and $Y'$ are homeomorphic. Then

$$\hat{X} \approx Y' \approx Y \approx \hat{X}'$$

so $X \approx X'$ also.

If $X$ and $X'$ are Möbius bands then we use Lemma 1.1. We have that $s(X)s(Y) = s(X')s(Y')$. Hence $s(Y) = s(Y')$, because $s(X) = s(X') = 1$. Since $H_1(Y) = H_1(Y')$ and $s(Y) = s(Y')$, they have the same number of components of their boundaries, so they are homeomorphic.

In the third case both surfaces $X$ and $Y$ are nonorientable. We cannot use exactly the same argument, but we make a similar consideration. First, we know by Lemma 1.2 that both surfaces $X'$ and $Y'$ are also nonorientable. We consider the manifolds $X \times S_j$ where $S_j$ are components of $\partial Y$, and $S_j \times Y$ where $S_j$ are components of $\partial X$.

Next, we take the oriented double covers $\hat{X}$ and $\hat{Y}$ of $X$ and $Y$. The manifolds $\hat{X} \times S_j$ and $S_j \times Y$ are the oriented double covers of $X \times S_j$ and $S_j \times Y$. Each of the tori $S_j \times S_i$ is covered by tori $S'_j \times S_i$ and $S''_j \times S_i$ in $\hat{X} \times S_i$ and is covered by tori $S'_j \times S''_i$ and $S_j \times S'_i$ in $S_j \times Y$.

By identifying $S'_j \times S_i$ with $S_j \times S'_i$ and $S''_j \times S_i$ with $S_j \times S''_i$, we obtain the oriented double cover $M$ of $\partial(X \times Y)$. It is not essential which circles we denoted by $S'_j, S''_j$, and $S'_i, S''_i$, because in every case we obtain the unique oriented double cover of $\partial(X \times Y)$.

Analogously, we construct the oriented double cover $M'$ of $\partial(X' \times Y')$. Of course $M$ and $M'$ are homeomorphic. If the manifolds $X$ and $Y$ are not the Möbius bands then we solve the problem by the Splitting Theorem.
If $X$ is a Möbius band then we solve the problem using Lemma 1.1, like in the second case. □

We also include the following new related result:

**Theorem 2.3** Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be any surfaces with nonempty boundary and suppose that their Cartesian products $X_1 \times \ldots \times X_n$ and $Y_1 \times \ldots \times Y_n$ are homeomorphic. Then there exists a one-to-one correspondence between them (assume $X_i$ corresponds to $Y_i$) such that $\text{rank} H_1(X_i) = \text{rank} H_1(Y_i)$ and if

$$s(X_i) = \text{rank} H_1(X_i) - \text{rank} H_1(\partial X_i) + 1$$

for $i = 1, 2, \ldots, n$ then

$$s(X_1) s(X_2) \ldots s(X_n) = s(Y_1) s(Y_2) \ldots s(Y_n).$$

**Proof.** Let $H_1(X_i) = Z^{m_i}$ and $H_1(Y_i) = Z^{m_i}$. We can conclude from the Künneth formula that

$$H_1(X_1 \times \ldots \times X_n) = Z^{\sum_{i=1}^n m_i};$$

$$H_2(X_1 \times \ldots \times X_n) = Z^{\sum_{i=1}^n n_i};$$

and

$$H_n(X_1 \times \ldots \times X_n) = Z^{n_1 \ldots n_n}.$$  

We obtain similar formulae for the product $Y_1 \times \ldots \times Y_n$. Because $\text{rank} H_i(X_1 \times \ldots \times X_n) = \text{rank} H_i(Y_1 \times \ldots \times Y_n)$ we can conclude that $n_i = m_i$ for $i = 1, 2, \ldots, n$. This follows from the fact that the ranks of the homology groups above are the coefficients of the polynomials $\prod_{i=1}^n (x - n_i)$ and $\prod_{i=1}^n (x - m_i)$. The polynomials are equal, so the numbers $n_i$ and $m_i$ are the same.

We obtain the equality $s(X_1) s(X_2) \ldots s(X_n) = s(Y_1) s(Y_2) \ldots s(Y_n)$ like in the previous proof. □

**Acknowledgements** The first and the second author were supported in part by the MESS program No. 101-500. The third author was supported in part by the UG grant No. BW 5100-5-0232-2 and the fourth author was supported in part by the UG grant No. BW 5100-5-0233-2. This research was also supported by the Polish-Slovenian grant No. SLO-POL-024 (2002-2003).
References


Jože Malešič, Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P. O. Box 2964, Ljubljana, Slovenia 1001, Email address: joze.malesic@uni-lj.si.

Dušan Repovš, Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P. O. Box 2964, Ljubljana, Slovenia 1001, Email address: dusan.repovs@uni-lj.si.

Wirotld Rosicki, Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, Gdańsk PL-80-952, Poland, Email address: wrosicki@math.univ.gda.pl.

Andreas Zastrow, Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, Gdańsk PL-80-952, Poland, Email address: matze@uni.gda.pl.