ON BASIC EMBEDDINGS INTO
THE PLANE

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Abstract. An subset $K \subset \mathbb{R}^2$ is said to be basic if for each function $f: K \to \mathbb{R}$ there exist functions $g, h: \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. If all the three functions in this definition are assumed to be continuous (differentiable), then the embedding is $C^0$-basic ($C^1$-basic). This notion appeared in studies of Hilbert’s 13th problem on superpositions. We prove that if a graph is $C^0$-basically embeddable in the plane, then it is $C^1$-basically embeddable in the plane. In our proof we construct an explicit $C^1$-basic embedding and use the Skopenkov characterization of graphs $C^0$-basically embeddable in the plane. Our result is non-trivial because the plane contains graphs which are $C^0$-basic but not $C^1$-basic and graphs which are $C^1$-basic but not $C^0$-basic (Baran-Skopenkov). We also prove that given any integer $k \geq 0$, there is a subset of the plane which is $C^r$-basic for each $0 \leq r \leq k$ but not $C^r$-basic for each $k < r \leq \omega$.

1. Introduction

The notion of a basic embedding appeared implicitly in the Kolmogorov-Arnold solution of Hilbert’s 13th problem [4, 1, 5]. A compactum $K \subset \mathbb{R}^2$ is said to be basic, if for each continuous function $f: K \to \mathbb{R}$ there exist continuous functions $g, h: \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. This note is motivated by the following problem.

Problem. Replace in the definition of a basic embedding continuous functions by smooth functions (by Lipschitz, Hölder, analytic, etc. functions).

Find conditions on a compactum $K \subset \mathbb{R}^2$, under which $K$ is basically embedded into the plane in the smooth sense.

Find conditions on a finite graph $K$, under which $K$ is basically embeddable into the plane in the smooth sense.

For a subset $K$ of the plane (not necessarily open) a function $f: K \to \mathbb{R}$ is said to be $r$-analytic, $0 \leq r < \infty$, if for each point $(x_0, y_0) \in K$ there exist

$$\{a_{ij}^r\}_{i,j=0}^r \subset \mathbb{R}$$

such that $a_{00} = f(x_0, y_0)$

and

$$f(x_0 + x, y_0 + y) = \sum_{i,j=0}^r a_{ij}^r x^i y^j + o((|x| + |y|)^r),$$

where $(x_0 + x, y_0 + y) \in K$ and $(x, y) \to (0, 0)$. Since $\mathbb{R} \subset \mathbb{R}^2$, this definition applies to functions $\mathbb{R} \to \mathbb{R}$ as well. Note that $0$-analytic is the same as continuous, 1-analytic

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for functions $\mathbb{R} \to \mathbb{R}$ is the same as differentiable and $r$-analytic for functions $\mathbb{R} \to \mathbb{R}$ is approximately (but not precisely) the same as $C^r$.

For a subset $K$ of the plane (not necessarily open) a function $f: K \to \mathbb{R}$ is said to be analytic (or $\omega$-analytic), if for each point $(x_0, y_0) \in K$ there exist

$$\{a_{ij}\}_{i,j=0}^\infty \subset \mathbb{R} \quad \text{such that} \quad f(x_0 + x, y_0 + y) = \sum_{i,j=0}^\infty a_{ij}x^iy^j$$

for $(x_0 + x, y_0 + y)$ belonging to some neighborhood of $(x_0, y_0)$ in $K$.

A compactum $K \subset \mathbb{R}^2$ is said to be $C^r$-basic, $1 \leq r \leq \omega$, if for each $r$-analytic function $f: K \to \mathbb{R}$ there exist $r$-analytic functions $g, h: \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

The main result of this paper is the following.

**Theorem 1.1.** If a finite graph $K$ is $C^0$-basically embeddable into the plane, then $K$ is $C^1$-basically embeddable into the plane.

Theorem 1.1 is non-trivial because the plane contains graphs which are $C^1$-basic but not $C^0$-basic and graphs which are $C^1$-basic but not $C^0$-basic (Baran-Skopenkov). Denote by $[a]$ the integer part of $a$. As a subset of the plane which is $C^1$-basic but not $C^0$-basic we can take $\{(2^{-\lfloor \frac{a}{2} \rfloor}, 2^{-\lfloor \frac{a+1}{2} \rfloor}) \cup \{(0, 0)\}$.

It is then easy to construct in the plane a graph which is $C^1$-basic but not $C^0$-basic. Note that a very similar subset $\{(\lfloor \frac{a+1}{2} \rfloor^{-1/2}, \lfloor \frac{a}{2} \rfloor^{-1/2}) \cup \{(0, 0)\}$ of the plane is not $C^1$-basic. Let $V$ be the graph of the function $y = |x|$, $x \in [-1, 1]$. As a subset of the plane which is $C^0$-basic but not $C^1$-basic we can take $(V - (2, 0)) \cup (V + (2, 0))$. Note that the subset $V$ itself is $C^1$-basic.

**Example 1.2.** Given any integer $k \geq 0$, there is a subset of the plane which is $C^r$-basic for each $0 \leq r \leq k$ but not $C^r$-basic for each $k < r \leq \omega$.

In Example 1.2 we can take the graph $V_k$ of the function $y = |x|^k$, $x \in [-1, 1]$ for $k$ odd, and $W_{k+1} = (V_{k+1} - (2, 0)) \cup (V_{k+1} + (2, 0))$ for $k$ even.

In the proof of Theorem 1.1 we use the following result, answering the Sternfeld problem [12].

**Theorem 1.3.** [10, cf. 6, 7, 9, §5] For any finite graph $K \subset \mathbb{R}^2$ the following conditions are equivalent:

(C) $K$ is $C^0$-basically embeddable in $\mathbb{R}^2$;

(G) $K$ does not contain any of the following three graphs: a circle $S$, a pentod $P$ or a cross $C$ with branched ends;

(E) $K$ can be embedded in $R_n$ for some $n$.

Definition of the graphs $R_n$ is given in §2. Our proof of Theorem 1.1 is based on a construction of a $C^1$-basic embedding $R_n \subset \mathbb{R}^2$ (§2). We prove elementary that this embedding is also $C^0$-basic, which yields an elementary proof of Theorem 1.3 as explained in §3.

2. Proofs

Let us define inductively the graphs $R_n$ together with an embedding $R_n \to \mathbb{R}^2$. We embed $R_1$ into $[-10, 10] \times [-10, 10]$ as shown in Figure 1. Then we repeat the
procedure by embedding copies of $R_1$ into squares $A$, $B$ and $C$ shown in Figure 1 to get $R_2$. Note that the embedded $R_1$ into $B$ was mirrored over $\ell$ to get a connected $R_2$.

![Figure 1](image_url)

In general, the graph $R_n$ is constructed by embedding $R_{n-1}$ into appropriate small squares $A$, $B$, $C$ attached to $R_1$. The squares $A$, $B$ and $C$ have to be chosen carefully. Let $p_1: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $p_2: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote projections onto $x$ and $y$ axes. We require that $p_1(A)$, $p_1(B)$, $p_1(C)$, $p_1(T)$ are disjoint and $p_2(A)$, $p_2(B)$, $p_2(C)$, $p_2(T)$ are disjoint.

**Proof of Theorem 1.1.** The boundary in $R_n$ of any subgraph $K \subset R_n$ consists of a finite number of points. Hence any 1-analytic mapping $K \to \mathbb{R}$ can be extended to a 1-analytic mapping $K \to \mathbb{R}$. So it suffices to prove that $R_n$ is $C^1$-basic. We prove this by induction. Given a mapping $f: R_n \to \mathbb{R}$ we shall find functions $g, h: \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$. Then we shall show that we can obtain $g$ and $h$ to be 1-analytic (i.e. differentiable), when $f$ is 1-analytic.

Put $h(0) = 0$ and define $g(x) = f(x, 0)$ for every $x \in [0, 2]$. Extend $g$ to a function $g: [0, 10] \to \mathbb{R}$.

Note that for every $y \in [0, 6]$ there exist an unique $x_y = |y| \in [0, 10]$ such that $(x_y, y) \in R_1$. Therefore using $g$ and $f$ for $x \in [0, 10]$ we can define $h: [-10, 6] \to \mathbb{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Extend $h$ to $h: [-10, 10] \to \mathbb{R}$.

Note that for every $x \in [-10, 0]$ there exists a unique $y_x = -x$ such that $(x, y_x) \in R_1$. Therefore using $h$ we can define $g: [-10, 0] \to \mathbb{R}$ as $g(x) = f(x, -x) - h(-x)$. Finally, we extend $g$ and $h$ to $g, h: \mathbb{R} \to \mathbb{R}$.
Now let \( f : R_n \to \mathbb{R}, n > 1 \), be given. We put \( h(0) = 0 \) and define \( g(x) = f(x, 0) \) for every \( x \in [0, 2] \). As \( R_n \) is constructed by embedding \( R_{n-1} \) into appropriate small squares \( A, B, C \) attached to \( R_1 \), by inductive hypothesis there exist functions \( g', h' : \mathbb{R} \to \mathbb{R} \) such that \( f(x, y) = g'(x) + h'(y) \) on \( (x, y) \in (A \cup B \cup C) \cap R_n \). Hence we can extend \( g \) smoothly onto \([0, 10]\) so that \( g = g' \) on \( p_1(B \cup C) \). Using functions \( g \) and \( f \) for \( x \in [0, 10] \) we can define \( h : [-10, 6] \to \mathbb{R} \) as \( h(y) = f(|y|, y) - g(|y|) \). Then we extend \( h \) onto \([-10, 10]\) so that \( h = h' \) on \([7, 10]\). Using \( h \) we finally define \( g : [-10, 0] \to \mathbb{R} \) as \( g(x) = f(x, -x) - h(-x) \).

For \( n = 1 \), if \( f \) is 1-analytic, then it is clear that at each step the constructed functions \( g \) and \( h \) are differentiable except maybe at 0. So all the extensions can be chosen to be differentiable. Since \( f \) is 1-analytic at \((0, 0)\), it follows that there exist \( a, b \in \mathbb{R} \) such that

\[
 f(x, y) = f(0, 0) + ax + by + o(|x| + |y|), \quad \text{where } (x, y) \in R_1 \quad \text{and} \quad (x, y) \to (0, 0).
\]

We may assume that \( f(0, 0) = g(0) = h(0) = 0 \). Then according to the structure of \( R_1 \) one can write

\[
 \begin{align*}
 f(x, x) &= g(x) + h(x) \\
 f(x, -x) &= g(x) + h(-x) \\
 f(x, 0) &= g(x) \\
 f(-x, x) &= g(-x) + h(x)
\end{align*}
\]

so

\[
 \begin{align*}
 g(x) &= f(x, 0) \\
 h(x) &= f(x, x) - f(x, 0) \\
 h(-x) &= f(x, -x) - f(x, 0) \\
 g(-x) &= f(-x, x) - f(x, x) + f(x, 0)
\end{align*}
\]

for small \( x \geq 0 \). Hence

\[
 g(x) = ax + o(x) \quad \text{and} \quad g(-x) = -ax + bx - ax - bx + ax + o(x) = -ax + o(x)
\]
when $x \to +0$. So $g$ is differentiable at 0. Also,

$$h(x) = ax + bx - ax + o(x) = bx + o(x)$$

and

$$h(-x) = ax - bx - ax + o(x) = -bx + o(x)$$

when $x \to +0$. So $h$ is differentiable at 0.

Hence for $n > 1$, if $f$ is 1-analytic, then it is clear that at each step the constructed functions $g$ and $h$ are differentiable everywhere. So all the extensions can be chosen to be differentiable and thus the resulting functions are differentiable. □

An elementary proof of $(R) \Rightarrow (C)$ in Theorem 1.3. Analogously to the proof of Theorem 1.1 above. The reduction from $K$ to $R_n$ follows also by the Tietze-Uryshon Extension Theorem. We construct $g$ and $h$ from $f$ as above. From the construction it is clear that at each step the constructed functions $g$ and $h$ are continuous. So all the extensions can be chosen to be continuous and thus the resulting functions are continuous. □

Note that for each function $f : R_1 \to \mathbb{R}$ the functions $g, h : \mathbb{R} \to \mathbb{R}$ such that

$$f(x, y) = g(x) + h(y)$$

are uniquely defined by $f$ in a neighborhood of 0. Hence any such functions $g$ and $h$ are 0- or 1-analytic in a neighborhood of 0, if $f$ is 0- or 1-analytic. Surprisingly, this is false for $r$-analytic functions with $1 < r \leq \omega$: the subset $R_1 \subset \mathbb{R}^2$ is $C^1$-basic but not $C^r$-basic for each $1 < r \leq \omega$. This is proved analogously to Example 1.2 for $k$ odd.

Proof of Example 1.2 for $k$ odd. First we prove that $V = V_1$ is $C^1$-basic. Take a 1-analytic function $f : V \to \mathbb{R}$. Since $f$ is 1-analytic at $(0, 0)$, it follows that there exist $a, b \in \mathbb{R}$ such that

$$f(x, [x]) = f(0, 0) + ax + b|x| + o(|x| + |x|), \quad \text{where} \quad x \to 0.$$  

Take $h(y) = by$ and $g(x) = f(x, [x]) - h([x])$. Clearly, $h$ is 1-analytic (i.e. differentiable) and $g$ is 1-analytic outside 0. Since $g(x) = f(0, 0) + ax + o(x)$ when $x \to 0$, it follows that $g$ is 1-analytic also at 0.

Now we prove that $V_k$ is $C^r$-basic for each $0 \leq r \leq k$. Take an $r$-analytic function $f : V_k \to \mathbb{R}$. Since $f$ is $r$-analytic at $(0, 0)$, it follows that there exist $a_{ij} \in \mathbb{R}$ such that

$$a_{00} = f(0, 0) \quad \text{and} \quad f(x, [x]^k) = \sum_{i,j=0}^r a_{ij}x^i[x]^kj + o([x] + [x]^r), \quad \text{where} \quad x \to 0.$$  

Since

$$o([x] + [x]^r) = o_1(x^r), \quad \text{we have} \quad f(x, [x]^k) = a_{00} + a_{01}[x]^k + a_{10}x + \cdots + a_{0r}x^r + o_2(x^r).$$

Take $h(y) = a_{01}y$ and $g(x) = f(x, [x]^k) - h([x]^k)$. Clearly, $h$ is $r$-analytic and $g$ is $r$-analytic outside 0. We also have $g(x) = a_{00} + a_{10}x + \cdots + a_{0r}x^r + o_2(x^r)$ when $x \to 0$. So $g$ is $r$-analytic also at 0.

Next we prove that $V = V_1$ is not $C^r$-basic for each $1 < r \leq \omega$. Define an analytic function $f : V \to \mathbb{R}$ by $f(x, y) = xy$, where $y = |x|$. If $V$ is $C^r$-basic for some $r \geq 2$, then there are $r$-analytic functions

$$g, h : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad f(x, [x]) = x|x| = g(x) + h(|x|) \quad \text{for each} \quad x \in [0, 1].$$
Hence \( g(x) - g(-x) = 2x^2 \). But this is impossible because \( g \) is 2-analytic, hence
\[
g(x) = g(0) + ax + bx^2 + o(x^2) \quad \text{and so} \quad g(-x) = g(0) - ax + bx^2 + o(x^2) \quad \text{for} \quad x \to +0.
\]

At last we prove that \( V_k \) is not \( C^r \)-basic for \( k \) odd and each \( k < r \leq \omega \). Define an analytic function \( f : V_k \to \mathbb{R} \) by \( f(x, y) = xy \), where \( y = |x|^k \). If \( V \) is \( C^r \)-basic for some \( r > k \), then there are \( r \)-analytic functions
\[
g, h : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad f(x, |x|^k) = x|x|^k = g(x) + h(|x|^k) \quad \text{for each} \quad x \in [0, 1].
\]

Hence \( g(x) - g(-x) = 2x|x|^k \). But this is impossible for \( k \) odd because \( g \) is \((k+1)\)-analytic, hence
\[
g(x) = g_0 + g_1 x + \cdots + g_{k+1} x^{k+1} + o(x^{k+1}) \quad \text{and so} \quad g(-x) = g_0 - g_1 x + \cdots + g_{k+1} x^{k+1} + o(x^{k+1})
\]
for \( x \to +0 \). \( \square \)

Note that a function \( f(x, y) \) on the graph \( V \) is 1-analytic if and only if \( p(t) = f(t, |t|) \) is differentiable on \([-1; 0] \) and on \([0; 1] \).

**Proof of Example 1.2 for \( k \) even.** Let us prove that \( W_{k+1} \) is \( C^r \)-basic for each \( 0 \leq r \leq k \). Given an \( r \)-analytic function \( f : W_{k+1} \to \mathbb{R} \), take functions \( h(y) = 0 \) and \( g(x) = f(x, |x|^2 \text{sign } x^{k+1}) \). Clearly, \( h \) is \( r \)-analytic and \( f(x, y) = g(x) + h(y) \) for each \((x, y) \in W_{k+1} \). Since the function \( p(t) = |t|^{k+1} \) is \( k \)-analytic and \( r \leq k \), it follows that \( g \) is \( r \)-analytic.

Let us prove that \( W_{k+1} \) is not \( C^r \)-basic for \( k \) even and each \( k < r \leq \infty \). Define an analytic function \( f : W_{k+1} \to \mathbb{R} \) by \( f(x, y) = y \text{sign } x \). If \( W_{k+1} \) is \( C^r \)-basic, then there are \( r \)-analytic functions \( g \) and \( h \) such that \( f(x, y) = g(x) + h(y) \). For
\[
x \in [-1, 1] \quad \text{we have} \quad g(\pm 2 + x) + h(|x|^{k+1}) = f(\pm 2 + x, |x|^{k+1}) = \pm |x|^{k+1}.
\]
Since \( g \) is \((k+1)\)-analytic and \( k+1 \) is odd, it follows that \( \frac{dh}{dx}|_{x=0} = +1 \) and \( \frac{dh}{dx}|_{x=0} = -1 \), which is a contradiction. \( \square \)

### 3. The Sternfeld criterion

The proof of Theorem 1.3 in [10] was based on the solution of the Arnold problem [2]: find conditions on a compactum \( K \subset \mathbb{R}^2 \), under which \( K \) is \( C \)-basic. This problem was solved by Sternfeld [11, 12] (who was apparently unaware of [2]). In order to formulate the Sternfeld criterion, let us introduce some definitions. Let \( p_1 \) and \( p_2 \) be projections onto the coordinate axes in \( \mathbb{R}^2 \). For \( Z \subset \mathbb{R}^2 \) let
\[
E(Z) = \{ z \in Z : |Z \cap p_1^{-1}(p_1(z))| \geq 2 \quad \text{and} \quad |Z \cap p_2^{-1}(p_2(z))| \geq 2 \}.
\]
Set \( E^2(Z) = E(E(Z)) \), \( E^3(Z) = E(E(E(Z))) \), etc. An ordered sequence \( \{a_1, \ldots, a_n\} \subset \mathbb{R}^2 \) is called an \textit{array}, if for each \( i \) we have \( p_1(a_i) = p_1(a_{i+1}) \) for \( i \) even and \( p_2(a_i) = p_2(a_{i+1}) \) for \( i \) odd \((a_i \neq a_{i+1}, \text{but it is not required that all the points of an array should be distinct}).
Theorem 3.1. [11, 12] For any compactum $K \subset \mathbb{R}^2$ the following conditions are equivalent:
(B) the embedding $K \subset \mathbb{R}^2$ is basic;
(E) $E^n(K) = \emptyset$ for some $n$;
(A) $K$ does not contain any array of $n$ points for some $n$.

In this paper we prove Theorem 3.1 following [12] (we believe our exposition is clearer). One can see that the proof of Theorem 3.1 is non-elementary in a sense that it used the Banach Inverse Operator Theorem.

The proof of (E) $\iff$ (G) in Theorem 1.3 is elementary, cf. [3]. The proof of (C) $\Rightarrow$ (G) in Theorem 1.3 is elementary modulo the implication (B) $\Rightarrow$ (A) of Theorem 3.1 [10]. The latter implication has an elementary proof by [8]. The proof of (R) $\Rightarrow$ (C) in Theorem 1.3 used the non-elementary implication (E) $\Rightarrow$ (B) of Theorem 3.1 [10]. In this paper we give an elementary proof of (R) $\Rightarrow$ (C) in Theorem 1.3, which yields an elementary proof of the whole Theorem 1.3.

The Sternfeld proof of Theorem 3.1. First we prove the easy assertion (A) $\Rightarrow$ (E). Suppose to the contrary that $E^n(K) \not= \emptyset$. Take a point $a_0 \in E^n(K)$. Then there exist points $a_{n-1}, a_1 \in E^{n-1}(K)$ such that $p_1(a_{n-1}) = p_1(a_0)$ and $p_2(a_1) = p_2(a_0)$. Analogously, there exist points $a_{n-2}, a_2 \in E^{n-2}(K)$ such that $\{a_{n-2}, a_{n-1}, a_0, a_1, a_2\}$ is an array. Analogously we construct an array of $2n + 1$ points in $K$.

The proof of (E) $\Rightarrow$ (G) $\Rightarrow$ (A) is based on a reformulation of (B) terms of linear operators in functional spaces. Denote by $C(X)$ the space of continuous functions on $X$ with the norm $\|f\| = \sup \{|f(x)| : x \in X\}$. For a subset $K \subset I^2$ define the linear superposition operator

$$\varphi : C(I) \oplus C(I) \rightarrow C(K) \quad \text{by} \quad \varphi(g, h)(x, y) = g(x) + h(y).$$

Clearly, the embedding $K \subset I^2$ is basic if and only if $\varphi = \varphi_K$ is epimorphic. Denote by $C^*(X)$ the space of bounded linear functionals on $C(X)$ with the norm $\|\mu\| = \sup \{|\mu(f)| : f \in C(X), |f| = 1\}$. For a subset $K \subset I^2$ define the dual linear superposition operator

$$\varphi^* : C^*(K) \rightarrow C^*(I) \oplus C^*(I) \quad \text{by} \quad \varphi^*(\mu, h) = (\mu(g \circ p_1), \mu(h \circ p_2)).$$

Since $|\varphi^*\mu| \leq 2|\mu|$, it follows that $\varphi^*$ is bounded. By duality, $\varphi_K$ is epimorphic if and only if $\varphi^* = \varphi_K^*$ is monomorphic. By the Banach Inverse Operator Theorem, $\varphi^*$ is monomorphic if and only if

(Φ) there exist $\varepsilon > 0$ such that $|\varphi^*\mu| > \varepsilon|\mu|$ for each $\mu \in C^*(K)$

(because this condition ensures that $\text{im } \varphi^*$ is closed). Thus (B) $\iff$ (Φ). So it remains to prove (E) $\Rightarrow$ (Φ) $\Rightarrow$ (A).

First we prove (Φ) $\Rightarrow$ (A). If (A) is false, then for each $n$ there exists an array $\{a_1, \ldots, a_n\} \subset K$. Define a linear functional $\mu \in C^*(K)$ by $\mu(f) = \sum_{i=1}^n (-1)^i f(a_i)$. Then $|\mu| = n$ and $|\varphi^*\mu| \leq 4$. Hence (Φ) is false.

Now we prove (E) $\Rightarrow$ (Φ). We use the fact that $C^*(X)$ is the space of $\sigma$-additive regular real valued Borel measures (in the sequel - simply ‘measures’) on $X$. We have

$$\varphi^*\mu = (\mu_x, \mu_y), \quad \text{where} \quad \mu_x(U) = \mu(p_1^{-1}U) \quad \text{and} \quad \mu_y(U) = \mu(p_2^{-1}U).$$

If $\mu = \mu^+ - \mu^-$ is the decomposition of a measure $\mu$ to its positive and negative parts, then $|\mu| = \tilde{\mu}(X)$, where $\tilde{\mu} = \mu^+ + \mu^-$ is the absolute value of $\mu$. Let $D_x$ ($D_y$) be the set
of points of $K$ which are not shadowed by some other point of $K$ in $x$- ($y$-) direction. Take any measure $\mu$ on $K$ of the norm $1$.

If

$$E(K) = \emptyset, \text{ then } D_x \cup D_y = K, \text{ so } 1 = \tilde{\mu}(K) \leq \tilde{\mu}(D_x) + \tilde{\mu}(D_y).$$

Therefore without loss of generality, $\tilde{\mu}(D_x) \geq 1/2$. Since $p_1$ is injective over $D_x$, it follows that $|\mu_x| \geq 1/2$, thus $(\Phi)$ holds.

If

$$E(E(K)) = \emptyset, \text{ then } D_x \cup D_y = K - E(K), \text{ so } E(D_x \cup D_y) = \emptyset.$$ 

Therefore in the case when $\tilde{\mu}(E(K)) < 3/4$ we have $\tilde{\mu}(D_x \cup D_y) > 1/4$ and without loss of generality $\tilde{\mu}(D_x) > 1/8$. Then as above $|\mu_x| > 1/8$, thus $(\Phi)$ holds. In the case when $\tilde{\mu}(E(K)) \geq 3/4$ we have $\tilde{\mu}(K - E(K)) \leq 1/4$. By the case $E(K) = \emptyset$ above without loss of generality $\tilde{\mu}_x(p_1(E(K))) \geq \tilde{\mu}(E(K))/2$. Hence $|\mu_x| \geq 1/8$, thus $(\Phi)$ holds. The case of arbitrary $n$ is proved analogously. \(\square\)

We remark that not only some linear relation on $\text{im } \varphi_K$ can force it to be strictly less than $C(K)$. Or, in other words, $\varphi_K^*$ can be injective but not monomorphic. If an embedding $K \subset \mathbb{R}^2$ is basic, then we can prove that $\varphi^*$ is monomorphic without use of $\varphi$ as follows. Define a linear operator

$$\Psi : C^*(I) \oplus C^*(I) \to C^*(K) \text{ by } \Psi(\mu_x, \mu_y)(f) = \mu_x(g) + \mu_y(h),$$

where $g, h \in C(I)$ are such that $g(0) = 0$ and $f(x, y) = g(x) + h(y)$ for $(x, y) \in K$. Clearly, $\Psi \Phi = \text{id}$ and $\Psi$ is bounded, hence $\Phi$ is monomorphic.

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