ON LINKING OF COMPACT SETS

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Abstract

We introduce a property \( \mathcal{L} \) for a subset of a manifold which enables us to pass the geometric linking property from the manifold to this subset. We prove that cubes with handles \( M \) and \( N \) are linked if and only if subsets \( X \subset \text{Int} \, M \) and \( Y \subset \text{Int} \, N \) having property \( \mathcal{L} \) are linked. We present a criterion which shows us that many of known Cantor sets explicitly given by defining sequences have this property. As an application of the property \( \mathcal{L} \) we extend the theorem on rigid Cantor sets thus allowing slightly more complicated terms in their defining sequences.

Keywords: geometric linking, Cantor set, defining sequence, rigid Cantor set.

AMS classification: 57M30

1. Introduction

Let \( \mathbb{E}^n \) be an Euclidean \( n \)-dimensional space and \( A, B \subset \mathbb{E}^n \) disjoint closed subsets. We say that \( A \) and \( B \) are (geometrically) unlinked if there exists \((n-1)\)-dimensional sphere \( S \subset \mathbb{E}^n \) which separates \( A \) and \( B \). We say that \( A \) and \( B \) are (geometrically) linked if such sphere doesn’t exist. One usually proves that two sets are geometrically unlinked by explicitly constructing the separating sphere.

Suppose now we have manifolds \( M \) and \( N \) with subsets \( X \subset M \) and \( Y \subset N \). If \( X \) and \( Y \) are the cores of respective manifolds it is obvious that \( X \) and \( Y \) are geometrically linked if and only if \( M \) and \( N \) are geometrically linked.

We will introduce a property \( \mathcal{L} \) which enables us to prove that subsets \( X \subset M \) and \( Y \subset N \) having this property are linked if \( M \) and \( N \) are linked.

As a corollary we will extend the theorem on rigid Cantor sets in \( \mathbb{E}^3 \) allowing slightly more complicated terms in their defining sequences.

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2. Property $\mathcal{L}$

Let $M \subset \mathbb{E}^n$ be a compact $n$-manifold with boundary. We say that a closed subset $A \subset \text{Int } M$ has a property $\mathcal{L}$ in $M$, if for every $n$-disk $B \subset \mathbb{E}^n \setminus A$ and every open neighbourhood $U \subset M \setminus A$ of $\text{Fr } M$ there exist such $n$-disk $B' \subset \mathbb{E}^n$ that $B \setminus \text{Int } M = B' \setminus \text{Int } M$ and $B' \cap M \subset U$.

![Figure 1: The set $A$ has a property $\mathcal{L}$ in $M$](image)

**Assertion 1.** Let $M \subset \mathbb{E}^n$ be a manifold and $S \subset \text{Int } M$ (any) core for $M$. Then $S$ has a property $\mathcal{L}$ in $M$.

**Proof.** Let $B \subset \mathbb{E}^n \setminus S$ be a $n$-disk and $U \subset M \setminus S$ open neighbourhood of $\text{Fr } M$ in $M$. As $S$ is a core of $M$ there exists a homeomorphism $h = (h_1, h_2) : M \setminus S \to \text{Fr } M \times [0, 1)$ satisfying $h(x) = (x, 0)$ for $x \in \text{Fr } M$. Hence there exists $\tau \in (0, 1)$ such that $h^{-1}(\text{Fr } M \times [0, \tau)) \subset U$. The mapping $f : M \setminus S \to h^{-1}(\text{Fr } M \times [0, \tau))$ defined by

$$f(x) = h^{-1}(h_1(x), \tau \cdot h_2(x)),$$

is a homeomorphism which is identity on $\text{Fr } M$. Finally we define $B' \cap M$ to be $f(B \cap M)$ as $f(B \cap M) \subset U$. ■

**Theorem 2.** Let $M \subset \mathbb{E}^3$ be a cube with handles and $N \subset \text{Int } M$ be manifold which is finite union of cubes with handles. Let there exist a finite collection $\mathcal{D}$ of 2-disks in $\text{Int } M$ with pairwise disjoint boundaries satisfying the following conditions:
1. disks in $\mathcal{D}$ intersect transversally and for every disk $D \in \mathcal{D}$ there exists such component $N'$ of $N$ that $\text{Fr} D = D \cap N'$;
2. no three disks from $\mathcal{D}$ intersect;
3. for every two disks $D, E \in \mathcal{D}$ the set $D \cap E$ is connected (it may be empty);
4. for every disk $D \in \mathcal{D}$ the set $D \setminus \left( N \cup \bigcup_{E \in \mathcal{D} \setminus \{D\}} E \right)$ is simply connected;
5. the set $M \setminus \left( N \cup \bigcup_{D \in \mathcal{D}} D \right)$ is connected and
6. there exists a set $A \subset N \cup \bigcup_{D \in \mathcal{D}} D$ having property $\mathcal{L}$ in $M$.

Then $N$ has a property $\mathcal{L}$ in $M$.

**Proof.** Let us denote $|\mathcal{D}'| = \bigcup_{E \in \mathcal{D}} E$ for any subset $\mathcal{D}' \subset \mathcal{D}$ and $\mathcal{D}_D = \{E \in \mathcal{D} \setminus \{D\}; E \cap D \neq \emptyset\}$ for any $D \in \mathcal{D}$. Let $\text{Fr } \mathcal{D} = \bigcup_{D \in \mathcal{D}} \text{Fr } D$.

We shall modify that part of the disk $B$ which lies in $M$ that the modified disk will not intersect $A$. Hence using the property $\mathcal{L}$ for $A$ one can further modify the disk $B$ such that $B \cap M$ lies arbitrary close to $\text{Fr } M$.

Let $B \subset \mathbb{R}^3$ be a 3-disk disjoint to $N$. Using a small move in $\text{Int } M \setminus N$ we can assume that $\text{Fr } B$ intersects $\mathcal{D}$ transversally. Fix an arbitrary 2-disk $D \in \mathcal{D}$. The set $B \cap D \neq \emptyset$ is either empty or every its component is a disk with holes. As the set $D \setminus N'$ is connected every component of $\text{Int } D \cap N$ is a 2-disk.

For every $E \in \mathcal{D}_D$ the set $D \cap E$ is an arc whose one boundary point lies in some disk in $\text{Int } D \cap N$ and the other boundary point lies in $\text{Fr } D$. (Both boundary points of the arc $D \cap E$ can not simultaneously lie in $\text{Fr } D$ as $D \setminus N'$
is connected and similarly they can not simultaneously lie in $N$ as in this case
the set $E \setminus N^*$ will not be connected.) By assumption of the theorem no three
disks in $\mathcal{D}$ intersect and hence arcs in $[\mathcal{D}] \cap D$ are pairwise disjoint.

Let $J$ be an arbitrary circle in $D \cap \text{Fr} B$. The circle $J$ bounds a 2-disk
(say $D_J$) in $\text{Int} D$. If $D_J \cap N \neq \emptyset$ then there exists a 2-disk $E \in \mathcal{D}$ such
that the arc $D \cap E$ has one boundary point on $N \cap D_J$ (otherwise $D \setminus N^*$ is
not simply connected) and the other boundary point on $\text{Fr} D$. The circle $J$
bounds two 2-disks (say $B_J$ and $B'_J$) on $\text{Fr} B$ and both of them are disjoint
with $\text{Fr} E$. Hence the intersection number (in $\mathbb{R}^3$) of the circle $\text{Fr} E$ and the
2-sphere $B_J \cup D_J$ equals to 1 which is certainly impossible. Therefore no 2-disk
in $N \cap \text{Int} D$ lies in any circle in $D \cap \text{Fr} B$.

Let $E \in \mathcal{D}$ be an arbitrary disk. Then $E \cap D$ is an arc having both
boundary points outside $D_J$. If $E \cap D_J \neq \emptyset$ using lemma 1 one can find a
2-disk $E_J \subset D_J$ which boundary consists of two arcs: one of them lies in $J$,
the other one lies in $E \cap D$. Using a small isotopy having its support in some
small neighbourhood of $E_J$ in $M$ one can appropriately move $B$ to diminish
the number of arcs in the intersection of $E \cap D$ and $D_J$.

Hence after finitely many steps we end up with disjoint 1-spheres in $\text{Fr} B \cap D$
and 2-disks in $E \in \mathcal{D}$.

Now choose outermost (with respect to $D$) circle in $D \cap \text{Fr} B$ and denote
it by $K$. (The circle $K$ is not necessarily unique.) The circle $K$ bounds some
2-disk $D_K$. Using a twosided collar of $D$ in $M$ we enlarge $D_K$ to $D_K \times [-1,1]$.
As $M \setminus N^*$ is connected we can connect points $(x,1) \in \text{Fr} D_K \times \{1\}$ and
$(x,-1) \in \text{Fr} D_K \times \{-1\}$ with some arc $w \subset M \setminus N^*$.

Let $W = w \times B^2$ be a small tubular neighbourhood of $w$ in $M \setminus N^*$.
Obviously $W \approx D_K \times [-1,1]$ and using an appropriate modification of $W$
near $\text{Fr} w \times B^2$ one can obtain $W \cap B \subset D_K \times [-1,1]$ and $\text{Fr} w \times B^2 = D_K \times \{-1,1\}$.

Hence we can divert that part of $B$ which lies in $D_K \times [-1,1]$ to $W$. (The
choice of outermost component of $D \cap \text{Fr} B$ was neccessary here. It is possible
though that $B \cap (D_K \times [-1,1])$ has more than one component.) We repeat
the procedure for other circles in $D \cap \text{Fr} B$ starting again with outermost ones.

We repeat the procedure for other 2-disks in $\mathcal{D}$. We end up with 3-disk
$B$ satisfying $\text{Fr} B \cap N^* = \emptyset$. As $A \subset N^*$ has a property $L$ we can modify $B$
accordingly.

We have used the following observation:

**Lemma 1.** Let $D$ be a 2-disk and $T$ a nonempty finite collection of pairwise
disjoint arcs in $D$ properly embedded in $D$. The boundaries of arcs in $T$
divide $\text{Fr} D$ into collection $L$ of circular arcs. Then there exist a disk $E \subset D$ bounded
by an arc from $T$ and an arc from $L$. 

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Figure 2: Modification of $B$ near $D_J$

**Proof.** We induct on number $n$ of arcs in $T$. The case $n = 1$ is obvious. Now let there be $n + 1$ arcs in $T$ and $t$ one of them. Using the inductive hypothesis there exist a 2-disk $E'$ bounded by $t' \in T$ and $l \in L$ for some $t'$ and $l$. If $t \cap E = \emptyset$ then $E = E'$ else $\text{Fr } t$ splits $l$ in three arcs. In this case the disk $E$ is bounded by one of them and $t$.

**Theorem 3.** Let $M$ and $N$ be cubes with handles in $\mathbb{E}^3$ and $X \subset \text{Int } M$ and $Y \subset \text{Int } N$ closed subsets having property $\mathcal{L}$. Then $X$ and $Y$ are geometrically linked if and only if $M$ and $N$ are geometrically linked.

**Proof.** It is obvious that a separating sphere for $M$ and $N$ separates $X$ and $Y$ too. Now assume to the contrary that there exists a separating sphere $S \subset \mathbb{E}^3$ for $X$ in $Y$ while $M$ and $N$ are geometrically linked. Due to the symmetry we may assume that $X \subset \text{Int } S$ and $Y \subset \text{Ext } S$. We may also assume that $S$ intersects $\text{Fr } M$ and $\text{Fr } N$ transversally. Let $B \subset \mathbb{E}^3$ be a 3-disk bounded by $S$.

If $B \cap N \neq \emptyset$ we can use property $\mathcal{L}$ for $Y$ in $N$ to replace $B$ with $B'$ which intersects $N$ near $\text{Fr } N$. Then using a small move near $\text{Fr } N$ we push $B'$ off $N$ to obtain $B''$. Due to simplicity we denote the 3-disk $B''$ with $B$ again. Then $X \subset \text{Int } B$ and $B \cap N = \emptyset$.
If $S \cap \text{Fr} M \neq \emptyset$ we embed $\mathbb{E}^3$ in $S^3 \equiv \mathbb{E}^3 \cup \{\infty\}$ naturally (one-point compactification). Choose an arbitrary point $b \in \text{Fr} B$. Then there exist an arc $J$ in $S^3 \setminus N$ connecting $b$ and $\infty$ which (except $b$) lies in $S^3 \setminus B$.

For some small regular neighbourhood $N(J)$ of an arc $J$ in $S^3$ the manifold $N(J) \cup B$ is a 3-disk disjoint to $N$. We note that $X \subset N(J) \cup B$.

The complement of $N(J) \cup B$ in $S^3$ is a 3-disk $B'$ disjoint to $X$. As $\infty \in N(J) \cup B$ we use the property $\mathcal{L}$ of $X$ to push the 3-disk $B'$ off $M$. So we have obtained a 3-disk $B''$ whose boundary is a sphere $S'$ separating $X$ in $Y$ and disjoint to $\text{Fr} M \cup \text{Fr} N$. Note that $Y \subset \text{Int} B'$ and $X \cap B' = \emptyset$. Let us simplify the notation again and denote $B''$ simply by $B$.

As $M$ and $N$ are geometrically linked we have $B \subset \text{Int} M$ (i.e. $M \subset B$ is not possible). The manifold $M$ is a cube with at least one handle because it is linked to $N$. Therefore there exists a properly embedded 2-disk $D$ in $M$ (i.e. $\text{Fr} D = D \cap \text{Fr} M$) such that $\text{Fr} D \not\subset 0$ in $\mathbb{E}^3 \setminus \text{Int} M$. We may assume that $D \cap B = \emptyset$. A small regular neighbourhood $N(D)$ of $D$ in $\mathbb{E}^3$ is a 3-disk which can be pushed off $M$ using property $\mathcal{L}$. This contradicts the fact that $\text{Fr} D \not\subset 0$ in $\mathbb{E}^3 \setminus \text{Int} M$.

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**Remark.** Recall that $A \subseteq M$ is geometrically central in a manifold $M$ if for any 2-disk with holes $H$ and any interior essential mapping $f: H \to M$ we have $f(H) \cap A \neq \emptyset$. In other words: if $f(H) \cap A = \emptyset$ then $f: H \to M$ is interior inessential and hence there exists a map $g: H \to \text{Fr} M$ which coincides with $f$ on $\text{Fr} H$.

It is interesting to note, that one can prove a theorem similar to the theorem 2 replacing property $\mathcal{L}$ with geometric centrality. However it is not known yet whether the following linking theorem similar to the theorem 3 is valid.

**Conjecture 1.** Let $M$ and $N$ be cubes with handles in $\mathbb{E}^3$ and $X \subset \text{Int} M$ and $Y \subset \text{Int} N$ closed subsets being geometrically central in $M$ resp. $N$. Then $X$ and $Y$ are geometrically linked if and only if $M$ and $N$ are geometrically central.

\[\]

**Definition 1.** A defining sequence $(M_i)_i \in \mathcal{D}(X)$ for a Cantor set $X \subset \mathbb{E}^3$ which consists of cubes with handles has a property $\mathcal{L}$ if for every $i$ and every component $M$ of $M_i$ the manifold $M \cap M_{i+1}$ has a property $\mathcal{L}$ in $M$.

**Theorem 4.** Let a defining sequence $(M_i)_i \in \mathcal{D}(X)$ for a Cantor set $X \subset \mathbb{E}^3$ have property $\mathcal{L}$. Then for every $i$ and for every component $M$ of manifold $M_i$ the Cantor set $X \cap M$ has property $\mathcal{L}$ in $M$. 

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PROOF. Let $X' = X \cap M$. Let $B$ be a 3-disk disjoint to $B$. First we prove that there exists a 3-disk $D$ such that $D \setminus \text{Int } M = B \setminus \text{Int } M$ and $D \cap M \cap M_{i+1} = \emptyset$. Among all 3-disks $D$ satisfying $D \setminus \text{Int } M = B \setminus \text{Int } M$ and $D \cap X' = \emptyset$ we choose such 3-disk that the number $j$, $j \geq i$, $D \cap M \cap M_j \neq \emptyset$ and $D \cap M \cap M_{j+1} = \emptyset$ is minimal. If $j > i$ we use property $\mathcal{L}$ to push $D$ out of every component of $M \cap M_{j+1}$ which contradicts the minimality of $j$. Hence $j = i$ and we use property $\mathcal{L}$ to move $D \cap M$ arbitrary close to $\text{Fr } M$. \hfill ■

3. Rigid Cantor sets

Definition 2. A defining sequence $(M_i)_{i \in \mathbb{Z}} \subseteq \mathcal{D}(X)$ for the Cantor set $X \subseteq \mathbb{R}^3$ is brittle if for every component $M$ of $M_i$ and for every component $M'$ of $M_{i+1} \cap M$ the following holds: if some loop in $\text{Fr } M$ is contractible in $M$ then this loop is contractible in $(M \setminus M_{i+1}) \cup M'$ as well.

A Cantor set which is endowed with a brittle defining sequence has some nice properties. The first and the second item of the following theorem can be proved as in [2, Lemma 5.6], the last item is a slight generalization of [5, Lemma 2.1].

Theorem 5. Let $(M_i)_{i=0}^\infty$ be a brittle defining sequence for a Cantor set $X \subseteq \mathbb{R}^3$, which consists of cubes with handles. Then:

1. For every nonempty subset $A \subseteq X$ every loop $J \subset M_1$ is contractible in $(M_1 \setminus X) \cup A$.
2. For every dense countable subset $A \subseteq X$ the set $(\mathbb{R}^3 \setminus X) \cup A$ is 1-ULC.
3. For every closed proper subset $A \subset X$ there exists a 3-disk $B \subset \text{Int } M_1$ such that $A \subset \text{Int } B$.

Definition 3. Let $A \subset \mathbb{R}^n$ be an arbitrary (closed) set. We say that the set $A$ is rigid, if for every homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^n$ it holds: if $f(A) = A$ then $f|_A = \text{id}_A$.

There are many examples of rigid sets. Martin [3] has constructed a rigid 2-sphere in $E^3$, Böthe [1] has constructed a simply connected curve in $E^3$. Wright [5] has constructed rigid Cantor set in $\mathbb{R}^3$ using Antoine necklaces and has later [4] generalized construction to $\mathbb{R}^n$, $n \geq 3$.

The key part of construction [5] is lemma [5, lemma 2.1]. If we substitute this lemma by theorem 3 we can take more general building blocks in the construction thus constructing many more rigid Cantor sets.

Let there be a defining sequence $(M_i)_{i=0}^\infty \in \mathcal{D}(X)$ for a Cantor set $X$. For every component $M$ of $M_i$ one can define a graph $\Gamma_{i}^M$ as follows:
• The components of \( M \cap M_{i+1} \) are the vertices of \( \Gamma_i^M \).

• There is a connection between \( M' \) and \( M'' \) in \( \Gamma_i^M \) if and only if \( M' \) and \( M'' \) are geometrically linked.

We say that \( \Gamma_i^M \) is a linking pattern of \( X \) in \( M \). Let

\[
\Gamma_i = \bigcup_{M \text{ component of } M_i} \Gamma_i^M,
\]

\[
\Gamma(X; (M_i)_i) = (\Gamma_0, \Gamma_1, \Gamma_2, \ldots).
\]

We say that \( \Gamma(X; (M_i)) \) is a linking pattern of \( X \) with respect to the defining sequence \( (M_i) \) or simply a linking pattern of \( X \).

**Lemma 2.** Let Cantor set \( X \) and \( Y \) be given by defining sequences \( (M_i)_{i=0}^\infty \in \mathcal{D}(X) \) and \( (N_i)_{i=0}^\infty \in \mathcal{D}(Y) \) such that:

1. both defining sequences have property \( \mathcal{L} \) (see definition 1);
2. both defining sequences are brittle (see definition 2);
3. for every component \( M \) of \( M_i \) the graph \( \Gamma_i^M \) is a cycle and for every component \( N \) of \( N_i \) the graph \( \Gamma_i^N \) is a cycle.

If \( h(X) \subset Y \) for some homeomorphism \( h: \mathbb{E}^3 \rightarrow \mathbb{E}^3 \) then there exist \( n \in \mathbb{N} \cup \{0\} \) and a component \( V \) of \( N_n \) such that \( h(X) = V \cap Y \) and \( \Gamma_i^{M_0} \cong \Gamma_i^V \).

**Proof.** This is essentially lemma [5, Lemma 3.1].

**Theorem 6.** Let Cantor set \( X \) be given by a defining sequence \( (M_i)_{i=0}^\infty \in \mathcal{D}(X) \) such that:

1. defining sequence has property \( \mathcal{L} \);
2. defining sequence is brittle;
3. for every component \( M \) of \( M_i \) the graph \( \Gamma_i^M \) is a cycle and
4. for every two different components \( M \) and \( N \) of \( M_i \) the sequences \( \Gamma(M \cap X) \) and \( \Gamma(N \cap X) \) are different.

Then \( X \) is a rigid Cantor set.

**Proof.** This is essentially [5, Theorem 3.2].

As it was implicitly proven in 2 the Cantor set which was used as building block in the construction was unsplittable (i.e. no two its points can be separated by a 2-sphere in its complement). Hence by the theorem 6 we get a rigid
Cantor set which complement has nontrivial fundamental group. So it is not possible to replace the building blocks in 2 with such ones with simply connected complement as these are completely splittable (ie. every two its points can by separated by a sphere in its complement).

In order to prove the following conjecture one has to find a different approach.

**Conjecture 2.** There exists a rigid Cantor set in $\mathbb{E}^3$ which complement has trivial fundamental group.

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