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CAYLEY GRAPHS

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PROPOSED RUNNING HEAD:

2-ARC-TRANSITIVE CAYLEY GRAPHS
Abstract

The classification of 2-arc-transitive Cayley graphs of cyclic groups, given in J. Alg. Combin., 5 (1996) by Alspach, Conder, Xu and the author, motivates the main theme of this article: the study of 2-arc-transitive Cayley graphs of dihedral groups. First, a previously unknown infinite family of such graphs, arising as covers of certain complete graphs, is presented, leading to an interesting property of Singer cycles in the group $PGL(2, q)$, $q$ an odd prime power, among others. Second, a structural reduction theorem for 2-arc-transitive Cayley graphs of dihedral groups is proved, putting us – modulo a possible existence of such graphs among regular cyclic covers over a small family of certain bipartite graphs of – a step away from a complete classification of such graphs. As a byproduct, a partial description of 2-arc-transitive Cayley graphs of abelian groups with at most 3 involutions is also obtained.

Keywords: permutation group, imprimitive group, abelian group, dihedral group, Cayley graph, 2-arc-transitive graph.
1 Introductory remarks

Throughout this paper groups are finite, and graphs are finite, simple, undirected and unless specified otherwise, connected. For the group-theoretic terminology not defined here we refer the reader to [3, 9, 12, 53].

For a graph $X$ we let $V(X)$, $E(X)$ and $A(X)$ denote the vertex set, the edge set, and the arc set of $X$ respectively. If $u$ and $v$ are adjacent (or neighbours) in $X$, we denote the corresponding edge by $[u,v]$ (or by its shorter version $uv$).

A permutation group $G$ is said to act semiregularly on a set $V$ if it has trivial point stabilizers. A transitive and semiregular group is said to be regular. Regular group actions are tied to the concept of Cayley graphs. Given a group $G$ and a generating set $Q = Q^{-1}$ of $G$ such that $1 \not\in M$, the Cayley graph $Cay(G, Q)$ of $G$ relative to $Q$ has vertex set $G$ and edges of the form $[g, gq], g \in G, q \in Q$. Clearly, the automorphism group of $Cay(G, Q)$ admits a left regular action of $G$. The converse is also true [45].

Adopting the terminology of Tutte [50], for $k \geq 0$, a $k$-arc in a graph $X$ is a sequence of $k + 1$ vertices $v_1, v_2, \ldots, v_{k+1}$ of $X$, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. Let $X$ be a graph and $G$ be a subgroup of its automorphism group $\text{Aut} X$. We say that $X$ is $(G, k)$-arc-transitive provided $G$ acts transitively on the set of $k$-arcs of $X$. In particular, when $G = \text{Aut} X$, we say that $X$ is $k$-arc-transitive and we say that it is exactly $k$-arc-transitive if it is $k$-arc-transitive but not $(k+1)$-arc-transitive. Also, 1-arc-transitive graphs and 0-arc-transitive graphs are usually referred to as arc-transitive and vertex-transitive, respectively.

When investigating the structure of graphs admitting transitive actions of subgroups of automorphisms one often relies on results which are purely group-theoretic in nature. On the other hand, a graph-theoretic language may often provide useful insights in the study of various problems in permutation groups. A particular case of such a fruitful interplay of group-theoretic and graph-theoretic concepts is dealt with in this paper and is motivated by the classical results on B-groups due to Schur and Wielandt (see Propositions 8.2 and 8.3) and by some of the more recent research in 2-arc-transitive Cayley graphs [1, 20]. (A group $G$ is a $B$-group if every subgroup containing a regular subgroup isomorphic to $G$ is either imprimitive or doubly transitive.)

Impelled by a complete classification of 2-arc-transitive Cayley graphs of cyclic groups in [1], the main purpose of this article is a thorough study of 2-arc-transitive Cayley graphs of dihedral groups. The fact that cyclic groups of composite order and dihedral groups are $B$-groups is of vital importance in this respect. Also, note that a vertex-transitive graph is 2-arc-transitive if and only if the restriction of a vertex stabilizer to the set of neighbours of the corresponding (fixed) vertex is a doubly transitive group (see [44, Lemma 9.4]). From a group-theoretic point of view, the objects dealt with in this article are therefore transitive permutation groups containing
a regular subgroup isomorphic to a dihedral group, and having point stabilizers acting doubly transitively on some nontrivial connected suborbit, that is, on some suborbit which gives rise to a connected orbital graph. The main result of this article is a structural reduction theorem for 2-arc-transitive Cayley graphs of dihedral groups which comes a step short from a complete classification of these graphs (as well as the corresponding group-theoretic counterparts) – see Theorem 2.1.

This article is organized as follows. In Section 2 we start by giving a brief history of the research on arc-transitive graphs and by laying out the basic strategy behind the analysis of 2-arc-transitive Cayley graphs of dihedral groups. We then present a previously unknown infinite family of 2-arc-transitive Cayley graphs of dihedral groups \( D_4(q+1), q \) an odd prime power, which are denoted by \( K_{q+1} \) and arise as certain 4-fold covers of the complete graph \( K_{q+1} \) (and 2-fold covers of \( K_{q+1,q+1} - |q + 1|K_2 \), the complete bipartite graph with a 1-factor removed). We close the section by phrasing Theorem 2.1. In Section 3 we introduce the concept of a symbol of a Cayley graph of a dihedral group, a useful encoding of the set of neighbours of a given fixed vertex in the graph. Section 4 is devoted to special terminology and notation pertaining to covers of graphs. In Section 5 we show that the graphs \( K_{q+1}^{4}, q \) an odd prime power, are indeed Cayley graphs of dihedral groups \( D_4(q+1) \), and give (as a consequence) an interesting property of Singer cycles in the group \( PGL(2,q) \). In Section 6 we obtain a partial classification of 2-arc-transitive Cayley graphs of abelian groups with at most three involutions. These results prove helpful in the investigation of two particular aspects of 2-arc-transitive Cayley graphs of dihedral groups: certain small length blocks of imprimitivity and the above mentioned symbols, a task carried out in Section 7. These results are then used in Section 8, where the proof of Theorem 2.1 is finally given.

Let us remark that, in contrast with some of the other work on 2-arc-transitive graphs \([2, 11, 21, 22, 30, 34, 35, 36]\), where the classification of finite simple groups is a vital and essential part, the underlying philosophy of this article stems from a pursuit of a different goal. With a blending of group-theoretic and combinatorial techniques and results from coverings of graphs we have, for the better part, managed to avoid the use of the classification of finite simple groups. We believe that having consciously opted for a lighter weaponry we have not only succeeded in making this article reasonably self-contained, but have also obtained a clearer insight into the intrinsic peculiarities of the mathematical objects dealt with.

2 Laying out the strategy

The study of arc-transitive graphs has its roots in the seminal paper of Tutte \([49]\), where it was proved that cubic graphs are at most 5-arc-transitive. A number of articles on the subject followed over the years (disguised at times in group-theoretic
language), most of them dealing with cubic and quartic arc-transitive graphs, others touching more general grounds (see for example [6, 10, 13, 14, 15, 17, 18, 19, 23, 24, 25, 27, 33, 46, 47, 50, 52, 54] for a by no means exhaustive list), and perhaps reaching its peak with [51], where using the classification of simple groups, Weiss extended Tutte’s bound to graphs of arbitrary valency by showing that no 8-arc-transitive graph exists. However, it was only much later that an infinite family of quartic 7-arc-transitive (finite) graphs was actually found [11].

An interest in this topic of research, in 2-arc-transitive graphs in particular, resurrected quite recently, together with an organized effort that has been set in motion to understand the structure of such graphs based on an analysis of their quasiprimitive and biquasiprimitive quotients. A word of explanation singling out the main feature that makes 2-arc-transitive graphs more tractable than their ”less symmetric” arc-transitive counterparts seems appropriate.

Given a transitive group $G$ acting on a set $V$, we say that a partition $\mathcal{P}$ of $V$ is $G$-invariant (or an imprimitivity block system of $G$) if the elements of $G$ permute the parts, that is, blocks of $\mathcal{P}$ block-wise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only $G$-invariant partitions of $V$, then $G$ is said to be primitive, and is said to be imprimitive otherwise. Clearly, the set of orbits of any normal subgroup of $G$ gives rise to a $G$-invariant partition of $V$: we call it a $G$-normal partition. In particular, if this subgroup is non-trivial and intransitive, then the corresponding partition is non-trivial and the group $G$ is imprimitive. Suppose that every $G$-normal partition of $V$ consists of at most two blocks. Then the group $G$ is said to be quasiprimitive if every $G$-normal partition is trivial, and is said to be biquasiprimitive if it has at least one $G$-normal partition with two blocks.

Now let $X$ be a $(G, 2)$-arc-transitive graph for some subgroup $G$ of $\text{Aut} \ X$ and let $\mathcal{P}$ be a $G$-invariant partition of the vertex set $V(X)$. Let $X_\mathcal{P}$ be the associated quotient graph of $X$ relative to $\mathcal{P}$, that is, the graph with vertex set $\mathcal{P}$ and edge set induced naturally by the edge set $E(X)$. If $|\mathcal{P}| \geq 3$, then the bipartite graphs induced by two adjacent blocks in $\mathcal{P}$ are all isomorphic and moreover, in view of 2-arc-transitivity of $G$, the blocks in $\mathcal{P}$ are independent sets and, moreover, any vertex $v \in V(X)$ has at most one neighbour in an arbitrary block in $\mathcal{P}$.

Let us apply this fact to the special case where $\mathcal{P}$ coincides with the set of orbits of a non-trivial normal subgroup $K$ of $G$. If $K$ has at least three orbits, then $X$ is a normal $K$-cover (since $K$ acts regularly, also referred to as a regular $K$-cover, a term we will be using in this article) of a $(G/K, 2)$-arc-transitive quotient graph $X_K = X_\mathcal{P}$. Moreover, assume also that $K$ is a maximal intransitive normal subgroup of $G$. Then every non-trivial normal subgroup of $G/K$ has at most two orbits and consequently $G/K$ is either quasiprimitive or biquasiprimitive. These observations were made first in [43] in a slightly more general context of locally primitive graphs, that is, vertex-transitive graphs with vertex stabilizers acting primitively on
the corresponding neighbours’ sets (see [44, Theorem 10.2]). They suggest an obvious (at least in principle) strategy leading to a possible classification of 2-arc-transitive graphs, involving the following two steps. Step 1 would be concerned with obtaining a description of quasiprimitive and biquasiprimitive 2-arc-transitive graphs, that is, graphs whose every 2-arc-transitive subgroup of automorphisms is either quasiprimitive or biquasiprimitive, in short, basic graphs. Step 2 would then consist in finding all 2-arc-transitive regular covers of basic 2-arc-transitive graphs.

It is Step 1 that has received much of the attention thus far, as part of a program for classifying quasiprimitive 2-arc-transitive graphs. In [30], Ivanov and Praeger have completed the classification of quasiprimitive 2-arc-transitive graphs of affine type, and Baddeley has given many constructions and a detailed description of quasiprimitive graphs of twisted wreath type [2]. More recently, Fang and Praeger have carried out a similar description of 2-arc-transitive graphs associated with the Suzuki groups and Ree groups [21, 22]. Finally, let us mention the three most recent papers by Li [34, 35, 36], where, among others, a classification of quasiprimitive 2-arc-transitive graphs of odd order and prime power order has been completed.

The importance of the concept of covers to various symmetry problems in graphs is well known and may be measured, among others, by the amount of research that has been done since the first published papers on the subject [16, 26, 28]. This makes the fact that Step 2 has remained almost unchallenged thus far, so much harder to understand. Part of the reason lies in certain combinatorial difficulties one encounters when faced with classification problems involving covers of graphs. Nevertheless, a few attempts have indeed been made, the most recent one perhaps in [20], where regular covers of complete graphs whose group of covering transformations is either cyclic or isomorphic to $\mathbb{Z}_p^2$, $p$ a prime, and whose fibre-preserving subgroup of automorphisms acts 2-arc-transitively, have been classified. Apart from the obvious canonical double covers, the list contains also certain regular $\mathbb{Z}_k$-covers and $\mathbb{Z}_p^2$-covers of complete graphs which, as will become apparent later on, are one of the essential ingredients of this paper. (Given a positive integer $n$, we shall use the symbol $\mathbb{Z}_n$ to denote the ring of residues modulo $n$ as well as the cyclic group of order $n$. This should cause no confusion.)

A second paper dealing, at least implicitly, with Step 2, is [1] where a complete classification of 2-arc-transitive Cayley graphs of cyclic groups is given. Following [1, Theorem 1.1] such a graph is one of the following: the cycle $C_n$, $n \geq 3$, which is $k$-arc-transitive for any $k \geq 2$; or the complete graph $K_n$, $n \geq 3$, which is exactly 2-arc-transitive; or the complete bipartite graph $K_{n/2,n/2}$, $n \geq 6$, which is exactly 3-arc-transitive; or $K_{n/2,n/2} - n/2K_2$, the complete bipartite graph minus a 1-factor, that is, the canonical double cover of $K_{n/2}$, where $n/2 \geq 5$ is odd, which is exactly 2-arc-transitive. The proof of this theorem is purely combinatorial and completely void of the concepts of basic graphs and regular covers. Nevertheless, it is worth men-
tioning that, translated into the language of this article, the above theorem says the following: first, complete graphs and complete bipartite graphs are, respectively, the only quasiprimitive and biquasiprimitive 2-arc-transitive circulants of valency greater than 2, and second, the only non-basic 2-arc-transitive circulants of valency greater than 2 are the canonical double covers \( K_{m,m} - mK_2, m \text{ odd} \), of complete graphs \( K_m \).

In this article we continue our investigation of the structure of 2-arc-transitive Cayley graphs by giving a partial extension of the above result to dihedral groups. Also, as a byproduct, we obtain a partial description of 2-arc-transitive Cayley graphs of abelian graphs with at most three involutions. (This is not surprising for the intersection of the classes of 2-arc-transitive Cayley graphs of dihedral groups and Cayley graphs of abelian graphs with at most three involutions is non-empty. For example, \( K_{2m,2m}, m \geq 2 \), is not only a circulant and a Cayley graph of the dihedral group \( D_{4m} \), but also a Cayley graph of the abelian group \( \mathbb{Z}_{2m} \times \mathbb{Z}_2 \) ) When applying the "2-steps" strategy for determining 2-arc-transitive graphs (as laid out above) to the special case of Cayley graphs of dihedral groups, we rely heavily on the fact that cyclic groups of composite order and all dihedral groups are \( B \)-groups (see Propositions 8.2 and 8.3). As a consequence, with the exception of complete graphs, all Cayley graphs of dihedral groups must have an imprimitive automorphism group. Moreover, as we shall see, it transpires that in the imprimitive case there always exists an imprimitivity block system associated with a non-trivial intransitive normal subgroup. This enables us to obtain a clean description of basic 2-arc-transitive Cayley graphs of dihedral groups. Apart from the complete graphs, the complete bipartite graphs, and cycles of order twice a prime number - which are also basic 2-arc-transitive circulants - the only new graphs are the incidence/nonincidence graphs of symmetric designs with a group of automorphisms acting doubly transitively on points and containing a regular cyclic subgroup. These are (as it may be deduced from [32, Theorem]: the incidence and nonincidence graphs \( B(PG(d, q)) \) and \( B'(PG(d, q)) \), respectively, associated with the projective spaces \( PG(d, q), d \geq 2 \), and the incidence and nonincidence graph \( B(H_{11}) \) and \( B'(H_{11}) \), respectively, of the unique Hadamard design \( H_{11} \) on 11 points.

In order to phrase the main result of this article, Theorem 2.1 below, we need to introduce the following additional notation. For an odd prime power \( q \), let \( V \) and \( V' \) denote two copies of the projective line \( PG(1, q) = GF(q) \cup \{ \infty \} \). Identify the vertex set of \( K_{q+1,q+1} - (q + 1)K_2 \) with \( V \cup V' \) in such a way that the edge set consists of all the edges of the form \([u, v'], u \neq v\). Then we let \( K_{q+1}^4 \) denote the regular \( \mathbb{Z}_2 \)-cover of \( K_{q+1,q+1} - (q + 1)K_2 \) where the voltage of the arc \((u, v')\) is 1 if and only if \( u, v \in GF(q) \) and \( u - v \) is a square in \( GF(q) \), and is 0 in all other instances. A more detailed discussion of this family of graphs (first discovered in [20], but defined in a less concise manner as certain 4-fold covers of \( K_{q+1} \)) is given in Section 5.

Recall that quasiprimitive and biquasiprimitive graphs are referred to as basic graphs.
Theorem 2.1. Let $n \geq 3$ and let $X$ be a connected, $2$-arc-transitive Cayley graph of a dihedral group of order $2n$. Then one of the following occurs.

(i) $X$ is a basic graph and is isomorphic to one of the following graphs: $C_{2n}$, $n$ a prime; $K_{2n}$; $K_{n,n}$; $B(H_{11})$ or $B'(H_{11})$ with $n = 11$; $B(PG(d,q))$ or $B'(PG(d,q))$, where $n = (q^d - 1)/(q - 1)$, $d \geq 2$ and $q$ is a prime power.

(ii) $X$ is not a basic graph and is isomorphic to $K_{n,n} - nK_2$ or to $K_{q+1}^4$, with $n = 2(q + 1)$ and $q$ an odd prime power.

(iii) $X$ is not a basic graph and is either a regular cyclic cover of a basic graph in (i) other than a cycle or a complete graph, or a regular cyclic cover of a non-basic graph in (ii).

In the next sections we carefully prepare the grounds for the proof of Theorem 2.1 (phrased in a slightly modified form as Theorem 8.4) which will be given in Section 8.

Finally, we remark that part (iii) of Theorem 2.1 can be further improved. This task is pursued in a sequel to this article [39].

3 Notation, terminology, examples

Let $X$ be a graph and let $u, v \in V(X)$. We let $N^i(u)$ denote the set of all vertices of $X$ at distance $i$ from $u$. More generally, we let $N^{i,j}(u,v) = N^i(u) \cap N^j(v)$. In particular $N(u,v) = N^{1,1}(u,v)$. For a subset $W$ of $V(X)$ we let $N(W)$ denote the set of all neighbours of vertices in $W$. For a pair of disjoint subsets $U$ and $W$ of $X$ we let $X[U,W]$, or just $[U,W]$, when the graph $X$ is clear from the context, denote the bipartite subgraph of $X$ with vertex set $U \sqcup W$ and edge set consisting of all edges of $X$ with one end-vertex in $U$ and the other in $W$.

Given a ring $\mathcal{R}$ we let $\mathcal{R}^\#$ and $\mathcal{R}^\dagger$ denote the nonzero elements of $\mathcal{R}$ and the group of invertible elements of $\mathcal{R}$, respectively. Moreover, for a subset $M$ of $\mathcal{R}$ let $M^\# = M \cap \mathcal{R}^\#$ denote the nonzero elements of $M$. Extending this notation to groups, we let $S^\# = S \setminus \{id\}$ denote the set of nontrivial elements in a subset $S$ of a group $G$. Also, we shall be using additive notation in the case of abelian groups.

For integers $m \geq 1$ and $n \geq 2$, let $M(m,n)$ denote the set of all $m \times m$-matrices $\mathbf{S}$ whose $(i,j)$-entry $S_{i,j} = -S_{j,i}$ is a subset of $\mathbb{Z}_n$ if $i \neq j$ and a subset of $\mathbb{Z}_n^\#$ if $i = j$. To each graph $X$ with an $(m,n)$-semiregular automorphism $\rho$, that is, an automorphism with $m$ orbits $W_i$, $i \in \mathbb{Z}_m$, of length $n$, we may associate a member of $M(m,n)$ in the following way. For each $i \in \mathbb{Z}_n$ choose $w_i \in W_i$. We call the matrix $\mathbf{S} \in M(m,n)$, with the $(i,j)$-entry $s_{i,j} = \{s \in \mathbb{Z}_n : [w_i, \rho^s w_j] \in E(X)\}$, the symbol of $X$ relative to the $(m+1)$-tuple $(\rho, w_1, \ldots, w_m)$. Conversely, each matrix $\mathbf{S} \in M(m,n)$ is a symbol of some graph with an $(m,n)$-semiregular automorphism, namely the graph $X(\mathbf{S})$ with vertex set $\{w_{i,x} : i \in \mathbb{Z}_m, x \in \mathbb{Z}_n\}$ and edge set $\{[w_{i,x}, w_{j,y}] : y - x \in s_{i,j}\}$. 

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Two instances of the above situation are of particular interest to us. First, the case $m = 1$ gives rise to the so called circulants, that is, Cayley graphs of cyclic groups. For simplicity reasons we let the set $S = S_{1,1}$ be called a symbol of the circulant $X(\mathcal{S})$, that is, the Cayley graph $\text{Cay}(\mathbb{Z}_n, S)$, denoted by $\text{Cir}(\alpha, S)$. Observe that in this case the symbol depends solely on the automorphism $\rho$ and not on the choice of a particular vertex. Second, the case $m = 2$ with $S_{1,1} = S_{2,2}$ gives rise to Cayley graphs of a dihedral group $D_{2n}$, for the purpose of this article referred to as dihedrants. We let the pair $[S, T] = [S_{1,1}, S_{2,2}]$ be called a symbol of the dihedral $X(\mathcal{S})$; this graph will be denoted by $\text{Dih}(2n, S, T)$.

Throughout the rest of this article we let $U = W_1 = \{u_i = \rho^i w_1 : i \in \mathbb{Z}_n\}$ and $V = W_2 = \{v_i = \rho^i w_2 : i \in \mathbb{Z}_n\}$ denote the two orbits of the $(\mathbb{Z}_n)$-semiregular automorphism $\rho$ of $\text{Dih}(2n, S, T)$. Thus a regular dihedral group contained in $\text{Aut} \text{Dih}(2n, S, T)$ is generated by the automorphisms $\rho$ and $\tau$ mapping according to the rules:

$$\rho u_i = u_{i+1}, \quad \rho v_i = v_{i+1}, \quad i \in \mathbb{Z}_n,$$

and

$$\tau u_i = u_{-i}, \quad \tau v_i = u_{-i}, \quad i \in \mathbb{Z}_n.$$  \hspace{1cm} (1)

The proof of the following result is straightforward.

**Proposition 3.1.** Let $n \geq 2$ be an integer, let $S \subseteq \mathbb{Z}_n^\#$, $T \subseteq \mathbb{Z}_n$, $a \in \mathbb{Z}_n^\#$ and $b \in \mathbb{Z}_n$. Then $\text{Dih}(2n, S, T) \cong \text{Dih}(2n, aS, aT + b)$.

Let us give a brief description of some of the known 2-arc-transitive dihedrants in terms of their symbols. For example, the pair $[\emptyset, \{t, t'\}]$, $t, t' \in \mathbb{Z}_n$, is a symbol of the cycle $C_{2n}$, if and only if $t - t' \in \mathbb{Z}_n^*$. On the other hand, $[\mathbb{Z}_n^\#, \mathbb{Z}_n]$ is the (only) symbol of the complete graph $K_{2n}$.

The situation with the complete bipartite graph and the complete bipartite graph with a 1-factor removed is somewhat trickier. Clearly, $[\emptyset, \mathbb{Z}_n]$ and $[\emptyset, \mathbb{Z}_n \setminus \{k\}]$, $k \in \mathbb{Z}_n$, are the obvious symbols of $K_{n,n}$ and $K_{n,n} - nK_2$, respectively. Alternative symbols of the form $[S, T]$, $S \neq \emptyset$ occur when $n$ is even, as will be seen in Section 7, where we give a more detailed discussion of symbols of (2-arc-transitive) dihedrants. Therefore Proposition 3.1 gives only a sufficient condition for the isomorphism of two dihedrants.

As for the graphs associated with projective spaces and the Hadamard design $H_{11}$, the relative symbols are clearly of the form $[\emptyset, T]$, where $T$ is a corresponding cyclic difference set of $\mathbb{Z}_n$. For example, we may let $T = \{0, 1, 3\}$ in the case of the Heawood graph, the incidence graph $B(\text{PG}(3, 2))$ of the smallest projective space, and $T = \{1, 3, 4, 5, 9\}$ in the case of the incidence graph $B(H_{11})$.

4 Graph coverings

In this section we give a more formal definition of various concepts pertaining to graph covers. Let $X$ be a connected graph, and let $\mathcal{P}$ be a partition of $V(X)$
into independent sets of equal size $k \geq 2$. The quotient graph $X_P$ is the graph with vertex set $P$, and two vertices $P_1$ and $P_2$ of $X_P$ are adjacent if and only if there is at least one edge between a vertex of $P_1$ and a vertex of $P_2$ in $X$. We say that $X$ is an $k$-fold cover of $X_P$ if it has the property that $P_1P_2$ is an edge if and only if the subgraph $X[P_1,P_2]$ of $X$ is isomorphic to $kK_2$. (Note that in view of our assumptions all covers are non-trivial and connected.) In this case the quotient graph $X_P$ is called the base graph of $X$ and the sets $P_i$ are called the fibres of $X$. The subgroup $K$ of all those automorphisms of $X$ which fix each of the fibres setwise is called the group of covering transformations. Note that since $X$ is connected, the action of this group on the fibres of $X$ is necessarily semiregular. In particular, if this action is regular we say that $X$ is a regular $K$-cover or a normal $K$-cover (in short a regular cover or a normal cover) of $X_P$. (Note that 2-fold covers are necessarily regular covers.) In this case we can reconstruct the graph $X$ from the base graph $X_P$ via a voltage assignment $\zeta : A(X_P) \rightarrow K$, that is, a function from the set of arcs of $X_P$ into the group $K$ where reverse arcs carry inverse voltages. By $\zeta_{P_1,P_2}$ we denote the voltage assigned to the arc $(P_1,P_2) \in A(X_P)$. We identify the vertex set of $X$ with the set \{(P,g) : P \in P, g \in G\} and we have that $(P_1,g_1)$ and $(P_2,g_2)$ are adjacent if and only if $P_1$ and $P_2$ are adjacent in $X_P$ and $g_1^{-1}g_2 = \zeta_{P_1,P_2}$. Hence a base graph with a voltage assignment gives rise to a graph cover in a natural way (see also [29]). We shall use the notation $X = \text{Cov}(X_P,\zeta)$. Let us stress the double role of the group $K$ of covering transformations, reflected in its left regular action on the vertices of the fibres, and in the voltage multiplication done on the right.

The voltage assignment $\zeta$ naturally extends (by group multiplication) to walks in $X_P$. In particular, for any walk $W$ of $X_P$ we let $\zeta_W$ denote the voltage of $W$. By connectedness of $X$, the voltages of all fundamental closed walks at any vertex $P \in P$ generate the whole voltage group $G$. It is also well known [29] that a given voltage assignment can be modified so that the arcs of an arbitrarily prescribed spanning tree receive the trivial voltage, and that the modified assignment is associated with the same graph cover. Namely, the modified voltage of each cotree arc is precisely the voltage of the corresponding fundamental closed walk relative to a fixed chosen base vertex $P \in P$. Moreover, the following proposition holds.

**Proposition 4.1.** [48] Leaving the voltages of a spanning tree trivial and replacing the voltage assignments on the cotree arcs by their images under an automorphism of the voltage group results in a voltage assignment associated with the same graph cover.

A lift $\tilde{\alpha}$ of an automorphism $\alpha$ of the base graph $X_P$ is an automorphism of the cover $X$ which projects along the group of covering transformations $K$. Also, we call $\alpha$ a projection of $\zeta$. Concepts such as the lift of a group of automorphisms and the projection of a group of automorphisms are self-explanatory. In particular, the group of covering transformations $K$ is the lift of the trivial subgroup of $\text{Aut} X_P$. 

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The canonical double cover of a graph $Y$ is the graph obtained by assigning the voltage $1 \in \mathbb{Z}_2$ to every arc of $Y$. Note that the canonical double cover of $K_n$ is isomorphic to $K_{n,n} - nK_2$.

The next two propositions provide information about the relationship between automorphisms of graph covers and their base graphs. The first one is taken from [37, Corollary 4.3]), whereas the second one may be deduced from [38, Corollaries 9.4, 9.7, 9.8].

**Proposition 4.2.** Let $Y$ be a graph and let $X = \text{Cov}(Y, \zeta)$ be a regular cover of $Y$ with respect to the voltage assignment $\zeta$. Then an automorphism $\alpha$ of $Y$ lifts to an automorphism of $X$ if and only if for each closed walk $W$ in $Y$ the voltage $\zeta_{\alpha W}$ is trivial precisely when $\zeta_W$ is trivial.

**Proposition 4.3.** Let $X$ be a regular $K$-cover of $Y$. A group $G \leq \text{Aut}Y$, acting semiregularly on $V(Y)$, lifts as a direct product $G \times K$ if and only if there exists a voltage assignment $\zeta : A(Y) \to K$ such that for each $\alpha \in G$ and each walk $W$ of $Y$ we have that $\zeta_W = \zeta_{\alpha W}$.

5 The graphs $K^4_{q+1}$ and Singer cycles

In [20, Theorem 1.1] a classification of all regular cyclic covers and regular $\mathbb{Z}_p^2$-covers, $p$ a prime, of complete graphs, with a 2-arc-transitive group of covering transformations, was given. It was shown that, apart from the canonical double covers, the graphs $X_1(4,q)$ and $X_2(4,q)$, $q$ an odd prime power, defined below are the only other such graphs.

For a prime power $q$, let $F(q) = GF(q)$ be the Galois field of order $q$ and let $SF(q)$ and $NF(q)$ be, respectively, the set of all squares and the set of all non-squares in $F(q)^\ast$. Further, let $Z$ be a graph isomorphic to $K_{q+1}$ where the vertex set is identified with the projective line $PG(1,q) = F(q) \cup \{\infty\}$.

Let $q \equiv 3 \mod 4$. Note that $SF(q) \cap -SF(q) = \emptyset$, that is, $NF(q) = -SF(q)$. We define $X_1(4,q)$ to be the 4-fold cover $\text{Cov}(Z, \zeta)$, where the voltage assignment $\zeta : A(Z) \to \mathbb{Z}_4$ is given by the following rule.

$$\zeta(x,y) = \begin{cases} 0, & \in \{x,y\} \\ 1, & y - x \in SF(q) \\ 3, & y - x \in NF(q). \end{cases}$$

Suppose now that $q \equiv 1 \mod 4$. Note that $SF(q) = -SF(q)$. We define $X_2(4,q)$ to be the 4-fold cover $\text{Cov}(Z, \zeta)$, where the voltage assignment $\zeta : A(Z) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by the following rule.

$$\zeta(x,y) = \begin{cases} (0,0), & \in \{x,y\} \\ (1,0), & y - x \in SF(q) \\ (0,1), & y - x \in NF(q). \end{cases}$$
It was proved in [20, Proposition 3.2] that \( X_i(4,q) \), \( i = 1, 2 \), are Cayley graphs. However, the fact that the corresponding groups may be taken to be dihedral was overlooked. There is a more concise way of describing these graphs as covers of \( Y \cong K_{q+1,q+1} - (q + 1)K_2 \). For an odd prime power \( q \), let \( V \) and \( V' \) denote two copies of the projective line \( PG(1,q) \). Identify \( V(Y) \) with \( V \cup V' \) and let the edge set consist of all the edges of the form \([u,v']\), \( u \neq v \). Then we let \( K_{q+1}^1 \) be the regular \( \mathbb{Z}_2 \)-cover \( \text{Cov}(Y, \zeta) \) where the voltage assignment \( \zeta : A(Y) \to \mathbb{Z}_2 \) is given by the following rule.

\[
\zeta(x, y') = \begin{cases} 
0, & x \in \{x, y\} \\
0, & y - x \in \mathbb{F}(q) \\
1, & y - x \in \mathbb{F}(q).
\end{cases}
\] (5)

It may be easily seen that \( K_{q+1}^1 \) is isomorphic to \( X_1(4,q) \) when \( q \equiv 3 (\text{mod} 4) \) and to \( X_2(4,q) \) when \( q \equiv 1 (\text{mod} 4) \). For example, the corresponding isomorphism \( \varphi : X_2(4,q) \to K_{q+1}^1 \) in the case \( q \equiv 1 (\text{mod} 4) \) maps according to the following rules. (The case \( q \equiv 3 (\text{mod} 4) \) is done in a similar manner.)

\[
\varphi(\infty, (i,j)) = \begin{cases} 
(\infty', 0), & (i,j) = (0,0) \\
(\infty, 0), & (i,j) = (1,0) \\
(\infty, 1), & (i,j) = (0,1) \\
(\infty', 1), & (i,j) = (1,1).
\end{cases}
\]

and, for each \( x \in \mathbb{F}(q) \),

\[
\varphi(x, (i,j)) = \begin{cases} 
(x, 0), & (i,j) = (0,0) \\
(x', 0), & (i,j) = (1,0) \\
(x', 1), & (i,j) = (0,1) \\
(x, 1), & (i,j) = (1,1).
\end{cases}
\]

We now turn to the main aim of this section and give the proof that the graphs \( K_{q+1}^1 \) are indeed Cayley graphs of dihedral groups. Proposition 4.3 together with the two lemmas below will play a vital role in that respect.

**Lemma 5.1.** Let \( Y \cong K_{q+1,q+1} - (q + 1)K_2 \) with vertex set \( V(Y) = V \cup V' \) consisting of two disjoint copies of \( PG(1,q) \). Let \( u, v, w \in V \) be distinct vertices with \( v', v'', v'' \) as their counterparts in \( V' \) and let \( C = uv'v''w'uv' \) be the corresponding induced 6-cycle in \( Y \). Let \( \zeta \) be the voltage assignment on \( Y \) giving rise to \( K_{q+1}^1 = \text{Cov}(Y, \zeta) \). Then

\[
\zeta_C = \begin{cases} 
0, & q \equiv 1 (\text{mod} 4) \\
1, & q \equiv 3 (\text{mod} 4).
\end{cases}
\]

**Proof.** Observe first that, viewing \( Y \) as a canonical double cover of \( Z \cong K_{q+1}^1 \), and identifying \( V(Z) \) with \( V \), the 6-cycle \( C \) arises from the triangle \( T = uvwu \) in \( Z \).
Letting $\zeta'$ be the voltage assignment on $Z$ giving rise to $K_{q+1}^4 = \text{Cov}(Z, \zeta')$, we may see that the statement of Lemma 5.1 is equivalent to $\zeta'_T$ being an element of order 2 or 4 depending on whether $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$, which is clearly the case in view of (3) and (4). ■

The following generalisation of [1, Lemma 2.2], proved in [40, Lemma 2.1], will be quite consistently used throughout the rest of this paper.

**Lemma 5.2.** Let $A$ be an abelian group and $J$ be the set of involutions in $A$. Let $R \subseteq A^\# = A \setminus \{0\}$, let $a \in A^\#$ and set $\lambda_a = |R \cap (\pm R + a)|$. Then

(i) $a \not\in 2R$ implies that $\lambda_a$ is even;
(ii) $a = 2r$ for some $r \in R$ and $|R \cap (J + r)|$ is odd imply that $\lambda_a$ is even;
(iii) $a = 2r$ for some $r \in R$ and $|R \cap (J + r)|$ is even imply that $\lambda_a$ is odd.

For an even integer $n$, let $E_n$ and $\Phi_n$ denote the set $2\mathbb{Z}_n$ of all even and the set $1 + 2\mathbb{Z}_n$ of all odd elements in $\mathbb{Z}_n$, respectively. We are now ready to show that the graphs $K_{q+1}^4$ are dihedrants.

**Theorem 5.3.** Let $q \geq 3$ be an odd prime power. Then $K_{q+1}^4$ is a 2-arc-transitive Cayley graph of the dihedral group $D_{4(q+1)}$ having a symbol of the form $[0,T]$ for some $T \subseteq \mathbb{Z}_{q+1}$.

**Proof.** In view of the isomorphisms $K_{q+1}^4 \cong X_i[4,q]$, $i = 1,2$, the 2-arc-transitivity of the graphs $K_{q+1}^4$ follows from [20, Theorem 1.1]. It remains to be shown that these graphs are indeed dihedrants, a fact that went unnoticed in [20].

Viewing the graphs $K_{q+1}^4$ as 4-fold covers of $Z \cong K_{q+1}$, we note that the whole group $\text{PGL}(2,q)$ (as a subgroup of $\text{Aut} Z$) has a lift [20, p. 288]. (In fact the whole group $\text{PGL}(2,q)$ lifts, but we do not need this here.) Recalling the definition of $K_{q+1}^4$ as a regular $\mathbb{Z}_2$-cover of $Y \cong K_{q+1} - (q + 1)K_2$ in (5), let $V(Y) = V \cup V'$, where $V$ and $V'$ are two copies of $\text{PGL}(1,q)$. We may assume that $V$ and $V'$ are the two orbits of an isomorphic copy of $\text{PGL}(2,q)$ in the automorphism group of the "intermediate" cover $Y$.

By way of contradiction, assume that $X = K_{q+1}^4$ is not a Cayley graph of $D = D_{4(q+1)}$. Consider a Singer cycle $\sigma$ of order $q + 1$ in $\text{PGL}(2,q)$, with $V$ and $V'$ as its two orbits. Since the index 2 subgroup of the lifted group, fixing the two bipartition sets of $X$, contains a lift of $\text{PGL}(2,q)$ and thus a lift of a simple group $\text{PSL}(2,q)$, it must act faithfully on each part of the bipartition. Hence a lift $\tilde{\sigma}$ of $\sigma$ has either two orbits of length $2(q + 1)$ or four orbits of length $q + 1$. But a graph having an automorphism with two orbits of equal lengths which are independent sets of vertices is clearly a dihedrant. Therefore the former forces $X$ to be a Cayley graph of $D_{4(q+1)}$, and so we may assume that the latter occurs. Hence $\langle \tilde{\sigma} \rangle$ lifts as a direct product, that
is, $\langle \bar{\sigma} \rangle \cong \mathbb{Z}_{q+1} \times \mathbb{Z}_2$. In view of Proposition 4.3 there exists a relabelling of the vertex set in such a way that $V(Y) = \{v_i : i \in \mathbb{Z}_{q+1}\} \cup \{v'_i : i \in \mathbb{Z}_{q+1}\}$, and there are voltages $\zeta_i \in \mathbb{Z}_2$, for $i \in \mathbb{Z}_{q+1}^\#$, such that $K^1_{q+1} \cong \text{Cov}(Y, \zeta)$, where $\zeta_{v_i, v'_j} = \zeta_{j-i}$ for all distinct $i, j \in \mathbb{Z}_{q+1}^\#$.

Let $i \in \mathbb{Z}_{q+1}^\# \setminus \{(q + 1)/2\}$, let $j \in \mathbb{Z}_{q+1}^\# \setminus \{i\}$ and consider the 6-cycle $C_{i,j} = v_0 v_i v_j v'_i v'_j v_0$ in $Y$. Its voltage $\zeta_{C_{i,j}}$ is $\zeta_i - \zeta_{j-i} + \zeta_{j-i} + \zeta_{j-i} - \zeta_j$ which is, of course, equal to $\zeta_i + \zeta_{j-i} + \zeta_{j-i} + \zeta_{j-i} + \zeta_{j-i} + \zeta_j$. Setting $j = 2i$ we have that $\zeta_{C_{i,2i}} = \zeta_{2i} + \zeta_{-2i}$. Combining this with Lemma 5.1 we get that, for each $i \in \mathbb{Z}_{q+1}^\# \setminus \{(q + 1)/2\}$,

$$\zeta_{2i} + \zeta_{-2i} = \begin{cases} 0, & q \equiv 1 \pmod{4} \\ 1, & q \equiv 3 \pmod{4}. \end{cases}$$

(6)

The case $q \equiv 3 \pmod{4}$ is now immediate. Note that $(q + 1)/2$ is even, and so letting $i = (q + 1)/4$, we get $0 = \zeta_{(q+1)/2} + \zeta_{(q+1)/2} = \zeta_{(q+1)/2} + \zeta_{-(q+1)/2} = 1$, a contradiction.

We may therefore assume that $q \equiv 1 \pmod{4}$. Let both $i, j$ be odd. Then, in view of (6), we have $0 = \zeta_{C_{i,j}} = \zeta_i + \zeta_{j-i} + \zeta_{j-i} + \zeta_{j-i}$. Clearly, $(q + 1)/2$ is odd, and also $\zeta_{(q+1)/2} = 0$. Therefore letting $j = (q + 1)/2$, we get that, for each $i \in \mathbb{Z}_{q+1}^\#$,

$$\zeta_i + \zeta_{-i} = 0.$$  

(7)

Define now the set $M = \{i \in \mathbb{Z}_{q+1}^\# : \zeta_i = 0\}$ and let $M^c = \mathbb{Z}_{q+1}^\# \setminus M$ be its complement in $\mathbb{Z}_{q+1}^\#$. Since $X$ is 2-arc-transitive, any two vertices at distance 2 have the same number of common neighbours, say $\lambda$. Let us choose these two vertices from the fibres $\{(v_0, 0), (v_0, 1)\}$ and $\{(v_i, 0), (v_i, 1)\}$, respectively. Computing $|N((v_0, 0), (v_i, 0))|$ we get that, for each $i \in \mathbb{Z}_{q+1}^\#$,

$$|M \cap (M + i)| + |M^c \cap (M^c + i)| = \lambda.$$  

(8)

Similarly, computing $|N((v_0, 0), (v_i, 1))|$ we have, for each $i \in \mathbb{Z}_{q+1}^\#$,

$$|M \cap (M^c + i)| + |M^c \cap (M + i)| = \lambda,$$

which, since $M$ and $M^c$ are symmetric by (7), gives us, for each $i \in \mathbb{Z}_{q+1}^\#$,

$$2|M \cap (M^c + i)| = \lambda.$$  

(9)

Taking the sum over all $i \in \mathbb{Z}_{q+1}^\#$ in (8) and (9), we obtain $|M|^2 + |M^c|^2 = (\lambda + 1)q$ and $2|M||M^c| = \lambda q$. Consequently, $q^2 = (|M| + |M^c|)^2 = (2\lambda + 1)q$, and so

$$\lambda = (q - 1)/2.$$  

(10)
Moreover, $|M| - |M^c|^2 = q$. Let $x = |M|$. Then by computation, $x^2 - qx + q(q-1)/4 = 0$, giving us
\[ \{|M|, |M^c|\} = \left\{(q + \sqrt{q})/2, (q - \sqrt{q})/2\right\}. \tag{11} \]

Now observe that
\[ |M \cap (M + i)| + |M \cap (M^c + i)| = |M \cap \mathbb{Z}_{q+1}^* + i| = \begin{cases} |M| - 1, & i \in M \\ |M| - \lambda/2 - 1, & i \notin M. \end{cases} \]

Therefore
\[ |M \cap (M + i)| = \begin{cases} |M| - \lambda/2 - 1, & i \in M \\ |M| - \lambda/2, & i \notin M^c. \tag{12} \end{cases} \]

Let $M_1 = \{m \in M : m + (q + 1)/2 \in M\}$. Then $M_1 = M_1 + (q + 1)/2$ and $M_2 \cap (M_2 + (q + 1)/2) = \emptyset$ where $M_2 = M \setminus M_1$. Similarly, let $(M^c)_1 = \{m \in M^c : m + (q + 1)/2 \in M^c\}$ and $(M^c)_2 = M^c \setminus (M^c)_1$. Since $(q + 1)/2$ is the only involution in $\mathbb{Z}_{q+1}$, Lemma 5.2 implies that $|M \cap (M + i)|, i \in \mathbb{Z}_{q+1}^*$, is even if $i \in 2M_1 \cup (2M)^c$ and is odd if $i \in 2M_2$. Combining this fact with (12), we see that either $M \subseteq \mathbb{E}_{q+1}^#$ or $M^c \subseteq \mathbb{E}_{q+1}^#$.

Assume first that $M \subseteq \mathbb{E}_{q+1}^#$. In particular, as $(q+1)/2$ is odd, $M_2 = M$ and $M_1 = \emptyset$, and so $M \cap (M + (q+1)/2) = \emptyset$. Hence $(M^c)_1 = (\mathbb{E}_{q+1}^# \setminus M) \cup ((\mathbb{E}_{q+1}^# \setminus M) + (q+1)/2)$.

Letting $j = (q + 1)/2$ in (8), we get by (10),
\[ (q - 1)/2 = \lambda = |M \cap (M + (q + 1)/2)| + |M^c \cap (M^c + (q + 1)/2)| = |M^c \cap (M^c + (q + 1)/2)| = |(M^c)_1| = 2|\mathbb{E}_{q+1}^# \setminus M| = 2((q - 1)/2 - x). \]

It follows that $|M| = x = (q - 1)/4$ and so $|M^c| = (3q + 1)/4$. Comparing this with (11) we have $(q - 1)/4 = (q - \sqrt{q})/2$ and so $(\sqrt{q} - 1)^2 = 0$, forcing $q = 1$, a contradiction.

If we assume that $M^c \subseteq \mathbb{E}_{q+1}^#$, then the same argument (with the roles of $M$ and $M^c$ interchanged) gives us $|M^c| = (q - 1)/4$ and $|M| = (3q + 1)/4$, which again leads to a contradiction. This shows that $X = K_{q+1}^4$ is a dihedrant.

Finally, the fact that the symbol is really of the form claimed in the statement of this theorem is now immediate. Hence proof. \[ \blacksquare \]

Although there seems to be no obvious rule for constructing the sets $T$ in the symbols $[0, T]$ of the graphs $K_{q+1}^4$, an additional property of these sets will be proved in Section 7. In Corollary 7.6 we shall see that $T$ decomposes into two subsets $T_1$ and $T_2$, such that $T_1 = -T_1$ and $T_2 = -T_2 + (q + 1)/2$. The smallest four graphs $K_{q+1}^4, q = 3, 5, 7, 9$, of respective orders 16, 24, 32 and 40, are shown in Figure 1. Note that $K_{q+1}^4$ is the incidence graph of the Moebius-Kantor configuration (the generalized Petersen graph $GP(8, 3)$). The fact that this graph is a Cayley graph of the dihedral group $D_{16}$ has been overlooked in the Foster census [5].
Theorem 5.3 has an interesting consequence regarding the structure of Singer cycles in the group $PGL(2, q)$. Recall that $PGL(2, q)$ is sharply 3-transitive and therefore, for any $x \in \mathbb{F}(q)^*$, there is precisely one Singer cycle of the form $\sigma_x = (0 \infty x \ldots)$. We now prove the following fact about this cycle.

**Corollary 5.4.** Let $x \in \mathbb{F}(q)^*$ and let $y_x$ be the image of $x$ under the Singer cycle $\sigma_x = (0 \infty x \ldots)$ in $PGL(2, q)$. Then

$$x(y_x - x) \in \begin{cases} \mathbb{N}\mathbb{F}(q), & q \equiv 1 \pmod{4} \\ \mathbb{S}\mathbb{F}(q), & q \equiv 3 \pmod{4}. \end{cases}$$

(13)

**Proof.** The proof is done indirectly by tying (13) to the statement of Theorem 5.3. Note that the vertex set of $K_{q^2+1}$ is $(V \cup V') \times \mathbb{Z}_2$. With no loss of generality we may assume that there is a lift $\tilde{\sigma}_x$ of $\sigma_x$ which maps $(\infty, 0)$ to $(x, 0)$. Let $y = y_x$. Let us first show that

$$\tilde{\sigma}_x(0, 0) = \begin{cases} (\infty, 1), & x(y - x) \in \mathbb{N}\mathbb{F}(q) \\ (\infty, 0), & x(y - x) \in \mathbb{S}\mathbb{F}(q). \end{cases}$$

(14)

Clearly, $\tilde{\sigma}$ takes the neighbours of $(\infty, 0)$ to the neighbours of $(x, 0)$. But $N((\infty, 0)) = \{0', 0\} \cup (\mathbb{F}(q') \times \{0\}) \cup \mathbb{N}(\mathbb{F}(q') \times \{1\})$, and $N((x, 0)) = \{\infty', 0\} \cup ((\mathbb{S}\mathbb{F}(q) + x)' \times \{0\}) \cup ((\mathbb{N}\mathbb{F}(q) + x)' \times \{1\})$. In particular, we have that

$$\tilde{\sigma}_x(0', 0) = (\infty', 0),$$

(15)

and moreover $\tilde{\sigma}_x$ takes $(x', 0)$ to $(y', 0)$ when $y - x \in \mathbb{S}\mathbb{F}(q)$ and to $(y', 1)$ when $y - x \in \mathbb{N}\mathbb{F}(q)$. Assume that $y - x \in \mathbb{S}\mathbb{F}(q)$. As above, considering the neighbours of $(x', 0)$ and $(y', 0)$ we may deduce that $\tilde{\sigma}$ maps $(0, 0)$ to $(\infty, 0)$ for $x \in \mathbb{S}\mathbb{F}(q)$, and to $(\infty, 1)$ for $x \in \mathbb{N}\mathbb{F}(q)$. This proves (14) in the case $y - x \in \mathbb{S}\mathbb{F}(q)$. The case $y - x \in \mathbb{N}\mathbb{F}(q)$ is done in an analogous way.

Now, by (15), it follows that $\tilde{\sigma}_x$ maps $N((0', 0))$ to $N((\infty', 0))$ and so, in particular, $(z, 0)$ is mapped to $(\sigma z, 0)$ if $z \in -\mathbb{S}\mathbb{F}(q)$ and is mapped to $(\sigma z, 1)$ if $z \in -\mathbb{N}\mathbb{F}(q)$. It follows that, for $q \equiv 1 \pmod{4}$, the vertex $(z, 0)$ is mapped to $(\sigma z, 0)$ if $z \in \mathbb{S}\mathbb{F}(q)$ and to $(\sigma z, 1)$ if $z \in \mathbb{N}\mathbb{F}(q)$. Similarly, for $q \equiv 3 \pmod{4}$, the vertex $(z, 0)$ is mapped to $(\sigma z, 0)$ if $z \in \mathbb{N}\mathbb{F}(q)$ and to $(\sigma z, 1)$ if $z \in \mathbb{S}\mathbb{F}(q)$. In other words, the number of "switches" between layers 0 and 1, one makes with a continual application of $\tilde{\sigma}_x$ until the fibre $\{0, 0\}$ is reached, equals the cardinality of $\mathbb{S}\mathbb{F}(q)$ if $q \equiv 1 \pmod{4}$, and the cardinality of $\mathbb{S}\mathbb{F}(q)$ if $q \equiv 3 \pmod{4}$. In short, the number of switches is $(q - 1)/2$, which is even for $q \equiv 1 \pmod{4}$ and odd for $q \equiv 3 \pmod{4}$. Therefore

$$(\tilde{\sigma}_x)^0(\infty, 0) = \begin{cases} (0, 0), & q \equiv 1 \pmod{4} \\ (0, 1), & q \equiv 3 \pmod{4}. \end{cases}$$

(16)

By Theorem 5.3, it follows that $\sigma_x$ lifts as an automorphism with two orbits of length $2(q + 1)$ and so, in particular, $(\tilde{\sigma}_x)^{q+1} \neq 1$. Thus $(\tilde{\sigma}_x)^{q+1}(\infty, 0) = (\infty, 1)$. Combining this with (14) and (16) we get (13), as required. □
6 Abelian groups

In this section we explore the properties of 2-arc-transitive Cayley graphs of abelian groups having at most three involutions. We obtain a partial classification which will be needed in the subsequent sections in our analysis of 2-arc-transitive properties of dihedrants.

The next result is taken from [1].

**Proposition 6.1.** Let $X$ be a connected 2-arc-transitive graph and let $v \in V(X)$. Then the bipartite graph $X[N(v), N^2(v)]$ is biregular.

**Proof.** Let $v \in V(X)$. Since $X$ is 2-arc-transitive, it follows that both $N(v)$ and $N^2(v)$ are orbits of the vertex stabilizer $\text{Aut}(X)_v$. The result follows. ■

**Proposition 6.2.** Let $X$ be a 2-arc-transitive graph of valency $d \geq 3$ which is not a complete graph, and let $v \in V(X)$. If there exists a vertex $u \in N^2(v)$ such that $|N(u, v)| = d$ then $X \cong K_{d,d}$, and if there exists a vertex $u \in N^2(v)$ such that $|N(u, v)| = d - 1$ then $X \cong K_{d+1,d+1} - (d+1)K_2$.

**Proof.** Clearly, in view of 2-arc-transitivity, if any of the above two conditions is true for one vertex at distance 2 from $v$, then it holds uniformly for all vertices at distance 2 from $v$. Now, if each vertex $u \in N^2(v)$ has $d$ neighbours in $N(v)$ then clearly $X \cong K_{d,d}$. Suppose now that each vertex $u \in N^2(v)$ has $d - 1$ neighbours in $N(v)$. Then any two vertices in $N^2(v)$ must have a common neighbour in $N(v)$ and hence are not adjacent. The regularity of $X$ then implies that the set $(V(X) \setminus \{v\}) \cup N(v) \cup N^2(v))$ is not empty. Let $u \in N^3(v)$ and suppose $u$ has a neighbour $y \in N^2(u)$ and a neighbour $z \in N^2(u)$. Since $u$ is the sole element of $N(y) \setminus N(v)$, it follows that the 2-arc $(z, u, y)$ is contained on no 4-cycles. We conclude that there is a unique vertex in $V(X) \setminus ((\{v\} \cup N(v) \cup N^2(v))$. Clearly $X \cong K_{d+1,d+1} - (d+1)K_2$. ■

We now turn to Cayley graphs of abelian groups. Recall that for an abelian group $A$, the set $A \setminus \{0\}$ of its nontrivial elements is denoted by $A^\#$.

**Proposition 6.3.** Let $A$ be an abelian group and let $R = -R$ be a subset of $A^\#$ such that the Cayley graph $X = \text{Cay}(A, R)$ is 2-arc-transitive. If $X$ is neither a cycle nor a complete graph, then its girth is 4.

**Proof.** As $X$ is not a cycle, we can choose $r_1, r_2 \in R$ such that $r_2 \neq -r_1, -r_1$ and therefore $(0, r_1, r_1 + r_2, r_2, 0)$ is a 4-cycle in $X$. If $X$ contains a triangle then, by 2-arc-transitivity, all 2-arcs of $X$ are contained in a triangle, implying that $X$ is a complete graph, a contradiction. ■
Note that in a 2-arc-transitive graph $X$ the number of common neighbours of two vertices at distance 2 is constant. It will be denoted by $\lambda(X)$ or just $\lambda$ when the graph $X$ is clear from the context. In particular, let $X = \text{Cay}(A, R)$ be a 2-arc-transitive Cayley graph of an abelian group $A$ with $\lambda \geq 2$, that is, of girth 4. Then letting $x \in (R + R)^\#$, it follows that $x \in N^2(0)$. Of course, by the above comment, $\left| N(0, x) \right| = \lambda(X) = \lambda$. But $N(0, x) = R \cap (R + x)$ and so

$$\left| R \cap (R + x) \right| = \lambda \text{ for each } x \in (R + R)^\#.$$  \hspace{1cm} (17)

If $\lambda$ is odd we have the following general result.

**Proposition 6.4.** Let $A$ be an abelian group and let $R = -R$ be a subset of $A^\#$ such that $X = \text{Cay}(A, R)$ is a 2-arc-transitive Cayley graph of valency $d \geq 3$ which is not a complete graph. If $\lambda(X)$ is odd, then $X$ is isomorphic either to $K_{d, d}$ or to $K_{d+1, d+1} - (d + 1)K_2$.

**Proof.** Since $X$ is not a complete graph, Proposition 6.3 implies that the girth of $X$ is 4 and so $(R + R)^\# \subset N^2(0)$. Therefore, by (17), it follows that $\lambda = \left| R \cap (R + t) \right|$ is odd for any $t \in (R + R)^\#$. So, by Lemma 5.2, we have that $N^2(0) = (R + R)^\#$. Therefore, if $0 \in 2R$ then $N^2(0)$ has cardinality $d - 1$. Thus, by Proposition 6.1, it follows that each vertex in $N^2(0)$ is adjacent to $d$ vertices in $N(0)$ and so, by Proposition 6.2, we have $X \cong K_{d, d}$. If $0 \notin 2R$ then $\left| N^2(0) \right| = d$, and Proposition 6.1 implies that each vertex in $N^2(0)$ is adjacent to $d - 1$ vertices in $N(0)$. Hence, by Proposition 6.2, we have $X \cong K_{d+1, d+1} - (d + 1)K_2$. $\square$

**Proposition 6.5.** Let $A$ be an abelian group, $J$ be the set of involutions in $A$, and let $R = -R$ be a subset of $A^\#$ of cardinality $d \geq 3$ such that the Cayley graph $X = \text{Cay}(A, R)$ is 2-arc-transitive and not isomorphic to a complete graph. Then either $\left| J \cap N^2(0) \right| \geq 2$ or $X$ is isomorphic either to $K_{d, d}$ or to $K_{d+1, d+1} - (d + 1)K_2$.

**Proof.** By Proposition 6.3, the girth of $X$ is 4 and consequently $R \cap (R + R) = \emptyset$ and so $N^2(0) = (R + R)^\#$. The proposition clearly holds if $R \setminus J = \emptyset$. Also, by Proposition 6.4, we may assume that $\lambda(X)$ is even. Therefore, in view of Lemma 5.2, we have that $\left| R \cap (J + \tau) \right|$ is odd for each $\tau \in R \setminus J$, and consequently $J \cap N^2(0) = J \cap (R + R) \neq \emptyset$. Assume that $\tau$ is the only element in $J \cap N^2(0)$. By Lemma 5.2, each element in $R \setminus J$ is adjacent to $\tau$, that is, $N(0, \tau)$ contains $R \setminus J$. Also, for any two distinct $\tau_1, \tau_2 \in J \cap R$, we have $\tau_1 + \tau_2 = \tau \in J \cap N^2(0)$. This implies that $\left| J \cap R \right| \leq 2$. Now if $J \cap R = \emptyset$, then $\lambda(X) = \left| N(0, \tau) \right| = d$, and $X \cong K_{d, d}$ by Proposition 6.2. If $\left| J \cap R \right| = 1$, then $\lambda = \left| N(0, \tau) \right| \geq d - 1$ and, by Proposition 6.2, $X$ is isomorphic either to $K_{d, d}$ or to $K_{d+1, d+1} - (d + 1)K_2$.

Finally, if $J \cap R = \{ \tau_1, \tau_2 \}$, then $\tau = \tau_1 + \tau_2$ and so $N(\tau) = R$ and, by Proposition 6.2, $X \cong K_{d, d}$. Hence proof. $\square$
**Proposition 6.6.** Let $A$ be an abelian group with at most one involution, let $R = -R$ be a subset of $A^\#$ and let the Cayley graph $X = \text{Cay}(A, R)$ be 2-arc-transitive. Then $X$ is either a cycle, a complete graph, a complete bipartite graph or a complete bipartite graph with a 1-factor removed.

**Proof.** Assume that $X$ is neither a cycle nor a complete graph. Then its girth is 4 by Proposition 6.3. By Proposition 6.4 we can assume that $\lambda = \lambda(X)$ is even. Hence Lemma 5.2 and (17) together imply that $|R \cap (J + r)|$ is odd for each $r \in R \setminus J \neq \emptyset$, where $J$ is the set of involutions in $A$. In particular, this forces $J \neq \emptyset$. Let $J = \{r\}$. Since $X$ has girth 4, we cannot have $\tau \in R$. It follows that $N(0, \tau) = R$, for given any $r \in R$ there exists $r' \in R$ such that $\tau + r = r'$. By Proposition 6.2, it follows that $X \cong K_{\lambda, \lambda}$.

The next two propositions deal with the distribution of involutions in Cayley graphs of abelian groups, isomorphic either to the complete bipartite graphs or to the complete bipartite graphs with a 1-factor removed.

**Proposition 6.7.** Let $X \cong K_{n,n}$ be a Cayley graph of an abelian group $A$. Let $H \cong \mathbb{Z}_2^k$ be the largest elementary abelian 2-group contained in $A$, and $J = H^\#$ the set of involutions in $A$. Then one of the following occurs.

(i) $J \subseteq N^2(0)$; or

(ii) there exists a basis $B$ of $H$ such that $J \cap N(0) = \bigcup_{i \in \mathbb{Z}} (2i + 1)B$ and $J \cap N^2(0) = \bigcup_{i \in \mathbb{Z}} 2iB$.

**Proof.** Suppose that $J_1 = J \cap N(0) \neq \emptyset$ and let $B$ be the maximal linearly independent subset of $J_1$. We claim that $B$ is a basis of $H$. Assuming this is not the case, let $B'$ be a basis of $H$ containing $B$, let $\beta \in B$ and let $\beta' \in B' \setminus B$. Since $B$ is a maximal linearly independent subset of $J_1$, it follows that $\beta + \beta' \notin J_1$. So $\beta + \beta' \in N^2(0)$. But $X[N(0), N^2(0)] \cong K_{n,n-1}$ and so $\beta$ is adjacent to $\beta + \beta'$. Consequently, $\beta' \in N(0)$, a contradiction, showing that $B$ is indeed a basis of $H$. The rest of (ii) is now immediate.

**Proposition 6.8.** Let $X \cong K_{n,n} - nK_2$ be a Cayley graph of an abelian group $A$. Let $H \cong \mathbb{Z}_2^k$ be the largest elementary abelian 2-group contained in $A$, and $J = H^\#$ the set of involutions in $A$. Then the following statements hold.

(i) $|J \subseteq N(0)| = 2^{d-1} - 1 = |J \subseteq N^2(0)|$; and

(ii) $|J \cap N^3(0)| = 1$.

**Proof.** Let $v$ be the vertex of $X$ at distance 3 from $0$ and let $\alpha$ be the unique element of $A$ which takes $0$ to $v$. Since $\alpha$ preserves distance in $X$ we must have $\alpha v = 0$ and hence
\(\alpha \in J\), proving (ii). Let \(J_1 = J \cap N(0)\) and \(J_2 = J \cap N^2(0)\). Clearly \(\alpha\) interchanges \(J_1\) and \(J_2\) and hence \(J_1\) and \(J_2\) have equal cardinality, forcing (ii). \(\blacksquare\)

Letting \(Q_k\), \(k \geq 2\), denote the \(k\)-dimensional hypercube, we let \(FQ_k\) denote the so-called folded cube of dimension \(k\), obtained from \(Q_{k+1}\) by identifying pairs of antipodal vertices.

**Theorem 6.9.** Let \(A\) be an abelian group, \(J\) be the set of involutions in \(A\) such that \(|J| \leq 3\), and let \(R = -R\) be a subset of \(A^\#\) such that the Cayley graph \(X = \text{Cay}(A, R)\) is 2-arc-transitive. For each positive integer \(i\) let \(J_i = J \cap N^i(0)\) be the subset of involutions at distance \(i\) from 0. Then one of the following holds.

(i) \(X\) is either a cycle or a complete graph; or

(ii) \(X\) is a complete bipartite graph and either \(J = J_2\) or \(|J_1| = 2\) and \(|J_2| = 1\); or

(iii) \(X\) is a complete bipartite graph with a 1-factor removed and \(|J_i| = 1\) for each \(i \in \{1, 2, 3\}\); or

(iv) \(|J_1| = 1, A \cong \mathbb{Z}_2^4, R = \{(1, 0), (3, 0), (0, 1), (0, 3), (2, 2)\}\) and \(X \cong FQ_5\); or

(v) \(J_1 = \emptyset, |J_2| = 2\) and \(X\) is a 2-fold cover of a complete bipartite graph; or

(vi) \(|J_2| = 3\).

**Proof.** Let us assume that \(X\) is neither a cycle nor a complete graph. If \(X\) is a complete bipartite graph, then (ii) follows by Proposition 6.7, and if \(X\) is a complete bipartite graph with a 1-factor removed, then (iii) follows by Proposition 6.8. We may therefore assume the \(X\) is neither of the above graphs.

By Proposition 6.4, we may assume that \(\lambda = \lambda(X)\) is even and thus \(\lambda \geq 2\), and by Proposition 6.6 we may assume that \(|J| = 3\). Let \(J = \{\tau_1, \tau_2, \tau_3\}\). Also, by Proposition 6.5, we have \(|J_2| \geq 2\). We distinguish two different cases, depending on whether \(J_1\) is the empty set or not.

**Case 1.** \(J_1 \neq \emptyset\).

Without loss of generality \(J_1 = \{\tau_3\}\). Since \(\lambda\) is even, Lemma 5.2 implies that every noninvoluntary element of \(R\) must have precisely one involution as a neighbour. If one of \(\tau_1\) or \(\tau_2\) is adjacent to all of \(R \setminus \{\tau_3\}\), then by Proposition 6.2 we can deduce that the graph \(X\) is isomorphic to \(K_{\lambda+2,\lambda+2} - (\lambda + 2)K_2\). Therefore \(R\) decomposes into three (mutually disjoint) subsets: \(\{\tau_3\}\), \(R_1 = N(0, \tau_1)\) and \(R_2 = N(0, \tau_2)\). Clearly \(|R_1| = \lambda = |R_2|\).

We now show that \(\lambda = 2\). Observe that \(N(\tau_1) = R_1 \cup \{\tau_2\} \cup (R_2 + \tau_1)\). Therefore \(\tau_1\) and each \(\tau_2 \in R_2\) have \(\tau_2\) as a common neighbour and so must have \(\lambda - 1\) common neighbours in the set \(R_2 + \tau_1\), which implies that \(X[R_2, R_2 + \tau_1] \cong K_{\lambda, \lambda} - \lambda K_2\). By
symmetry, \( X[R_1, R_1 + \tau_2] \cong K_{\lambda, \lambda} - \lambda K_2 \). Hence any two vertices in \( R_1 \) have precisely \( \lambda - 2 \) common neighbours in \( R_1 + \tau_2 \) and their only other common neighbours are 0 and \( \tau_1 \). Similarly, any two vertices in \( R_2 \) have precisely \( \lambda - 2 \) common neighbours in \( S_2 + \tau_1 \) and their only other common neighbours are 0 and \( \tau_2 \). Choose any \( r_1 \in R_1 \) and any two distinct elements \( r_2, r_2' \in R_2 \). All three vertices have 0 as a common neighbour. Besides, the remaining \( \lambda - 1 \) common neighbours of \( r_1 \) and \( r_2 \) or \( r_1 \) and \( r_2' \) are not in the sets \( R_2 + \tau_1 \) and \( R_1 + \tau_2 \) and hence they must be contained in the set of \( \lambda \) neighbours of \( r_1 \) other than 0, \( \tau_1 \) and its neighbours in \( R_2 + \tau_1 \). But \( r_2 \) and \( r_2' \) cannot have a common neighbour in this set of \( \lambda \) vertices, because all of their common neighbours are in \( (R_2 + \tau_1) \cup \{\tau_2, 0\} \). It follows that \( \lambda = 2 \) is the only possibility. Clearly, \( R_1 = \{r_1, -r_1\} \) and \( R_2 = \{r_2, -r_2\} \) and we obtain a unique graph isomorphic to \( FQ_5 \), which is a Cayley graph of \( \mathbb{Z}_4^1 \) (as well as of \( \mathbb{Z}_2^1 \)) and is 2-arc-transitive but not 3-arc-transitive (with vertex stabilizer isomorphic to \( S_5 \)). Also, the set of generators is as claimed.

Case 2. \( J_1 = \emptyset \).

Recall that \( |J_2| \geq 2 \). We assume that \( J_2 = \{\tau_1, \tau_2\} \) and show that \( X \) is a 2-fold cover of a complete bipartite graph.

Since \( \lambda \) is even and since \( J_1 = \emptyset \) and \( |J_2| = 2 \), it follows by Lemma 5.2 that every element of \( R \) must have precisely one involution as a neighbour. Hence \( R \) is a disjoint union of sets \( R_1 = N(0, \tau_1) \) and \( R_2 = N(0, \tau_2) \). Also \( |R_1| = \lambda = |R_2| \). (Note that \( R_i = -R_i \).) Observe also that \( R_i + \tau_i = R_i \) for \( i = 1, 2 \). Define \( R_1 + R_2 = \{r_1 + r_2 : r_1 \in R_1, r_2 \in R_2\} \) and let \( R_1 + R_1 = \{r_1 + r_1' : r_1, r_1' \in R_1\} \setminus ((R_1 + R_2) \cup \{0, \tau_1\}) \) and let \( R_2 + R_2 = \{r_2 + r_2' : r_2, r_2' \in R_2\} \setminus ((R_1 + R_2) \cup \{0, \tau_2\}) \).

We will now show that \( X \) has diameter 4 and that \( N^4(0) = \{\tau_3\} \). This will then imply that \( X \) is a 2-fold cover of a complete bipartite graph.

First, observe that \( N(\tau_3) \) is a disjoint union of the sets \( R_1 + \tau_3 = R_1 + \tau_2 \) and \( R_2 + \tau_3 = R_2 + \tau_1 \). Therefore \( N(\tau_1, \tau_3) = R_2 + \tau_1 \) and \( N(\tau_2, \tau_3) = R_2 + \tau_2 \). Let us prove that \( (R_1 + \tau_2) \cup (R_2 + \tau_1) \subset N^3(0) \). By symmetry we need only consider the set \( R_1 + \tau_2 \). If there was a vertex \( v = r_1 + \tau_2 \) in \( (R_1 + \tau_2) \cap N^2(0) \), then we would have \( |N(0, v)| = \lambda \). But since the girth of \( X \) is 4, we must have \( N(0, v) = R_1 \) which implies that, in particular, \( r_1 \) is adjacent to \( r_1 + \tau_2 \) and so \( \tau_2 \in R \), a contradiction. Therefore \( \tau_i, i = 1, 2 \), has no neighbours in \( N^2(0) \) and hence by 2-arc-transitivity the subgraph induced by \( N^2(0) \) is the null-graph.

Next we will see that \( \tau_3 \in N^3(0) \). Suppose that \( \tau_3 \in N^3(0) \). Then without loss of generality \( \tau_1 + \tau_2 = \tau_3 = r_1 + r + r' \) where \( r_1 \in R_1 \) and \( r, r' \in R \). But then we have \( r_1' = \tau_1 - r_1 = r + r' + \tau_2 \), implying that \( \tau_2 \) has a neighbour in \( N^2(0) \), a contradiction. We conclude that \( \tau_3 \in N^4(0) \).

Our next step will be to show that \( N(R_1 + R_2) \) is contained in the union \( U \) of the four sets \( R_1, R_2, R_1 + \tau_2 \) and \( R_2 + \tau_1 \). To that end note that a translation by \( \tau_3 \) fixes the set \( R_1 + R_2 \) while interchanging the sets \( R_1 \) and \( R_1 + \tau_2 \) and the sets \( R_2 \) and \( R_2 + \tau_1 \).
But each vertex in $R_1 + R_2$ has $\lambda$ neighbours in $R$ and thus, since its image under the translation by $\tau_3$ is in $R_1 + R_2$, it also has $\lambda$ neighbours in $R + \tau_3$. Hence all of its neighbours are in $\mathcal{U}$. Now we show that also $N(R_1 + R_2)$ is contained in $\mathcal{U}$. If a vertex $v = r_1 + r_2' \in R_1 + R_1$ has a neighbour in $N(\tau_3) = (R_1 + \tau_2) \cup (R_2 + \tau_1)$, then it must have $\lambda$ neighbours in this set in view of the fact that $v \in N^2(\tau_3)$. But then $N(v)$ is contained in $\mathcal{U}$ as $v$ already has $\lambda$ neighbours in $R$. So assume that $v$ has no neighbours in $N(\tau_3)$ and consider the set $v + R_2$. Now, for each $r_2 \in R_2$, $v + r_2 = r_1 + (r_1' + r_2)$ is in $\mathcal{U}$ but cannot be in $R_1$ as this would imply that $v = r_1' + r_2$ for some $r_1' \in R_1$, which is not possible for $v \in R_1 + R_1$. Hence $v + R_2 = R_2$ and as $v$ has at least one neighbour in $R$, we have that $|N(0,v)| \geq \lambda + 1$, a contradiction. Similarly we can show that $N(R_2 + R_2)$ is contained in $\mathcal{U}$. It follows that $N^3(0) = (R_1 + \tau_2) \cup (R_2 + \tau_1)$.

Finally we will show that $N(R_1 + \tau_2)$ (and equivalently $N(R_2 + \tau_1)$) is contained in $N^2(0) \cup \{\tau_3\}$. If not then there exists a vertex $v \in N^{3j}(0, \tau_3)$ for $j \in \{3, 4\}$. But then $v + \tau_3 \in N^{2j}(0, \tau_3)$, contradicting the fact that every vertex in $N^2(0)$ is also in $N^2(\tau_3)$. We conclude that $N^4(0) = \{\tau_3\}$ as well as that the diameter of $X$ is 4.

As $X$ is vertex-transitive, $Aut X$ has blocks of imprimitivity $B = \{\{v, v + \tau_3\} : v \in V(X)\}$. Clearly, if $u$ is adjacent to $v$ then $u + \tau_3$ is adjacent to $v + \tau_3$ and so by 2-arc-transitivity of $X$ we must have that $X$ is a 2-fold cover of a 2-arc-transitive graph, say $Y$. But $|V(X)| = 8\lambda$ with $|N(0)| = 2\lambda = |N^3(0)|$, $|N^2(0)| = 4\lambda - 2$ and $|N^4(0)| = 1$. Also, $X$ is bipartite with one half of the bipartition consisting of all blocks of the form $\{r, r + \tau_3\}$, $r \in R$. Therefore $Y$ is a bipartite graph with $4\lambda$ vertices and valency $2\lambda$ and so must be isomorphic to $K_{2\lambda, 2\lambda}$. This completes the proof of Theorem 6.9.

An investigation of 2-arc-transitivity properties of Cayley graphs of abelian groups, leading to a generalisation of the results of this section is continued in [41].

7 Blocks and symbols of dihedrants

Throughout the rest of this article we shall be assuming the notation of Section 3. Letting $S = -S$ be a symmetric subset of $\mathbb{Z}_n^d$, $n \geq 3$, and $T$ be a subset of $\mathbb{Z}_n$, we let $X = Dih(2n, S, T)$ be the dihedral group with symbol $[S, T]$, that is, the graph with vertex set $U \cup V$, where $U = \{u_i : i \in \mathbb{Z}_n\}$ and $V = \{v_i : i \in \mathbb{Z}_n\}$, and edges of the form $u_iu_{i+s}, v_iv_{i+t}$, for all $i \in \mathbb{Z}_n$ and $s \in S$, and $u_iv_{i+t}$, for all $i \in \mathbb{Z}_n$ and $t \in T$. Also, we let $\rho$ and $\tau$ denote the generators of a regular dihedral group in $Aut Dih(2n, S, T)$ – see (1) and (2). In particular, $U$ and $V$ are the two orbits of $\rho$. We emphasize that these automorphisms are assumed to play the same role for an arbitrary dihedral group of order $2n$, even if (for simplicity reasons) the sets $S$ and $T$ are not explicitly mentioned in the definition.

As already mentioned in the introductory section, by a classic result of Wiendt, every dihedral group is a $B$-group (see Proposition 8.3). The full automorphism group
of a dihedrant is therefore necessarily imprimitive, unless the graph or its complement is a complete graph. In this section we explore the nature of the corresponding imprimitivity block systems.

Let $X$ be a dihedrant of order $2n$, which is not a complete graph or its complement. We shall say that a block $B$ of $\text{Aut} \ X$ is cyclic if there exists a vertex $w \in V(X)$ and $m \in \mathbb{Z}_n^\#$ such that $B$ coincides with the orbit $\langle \rho^m \rangle w$. Further, we shall say that a block $B$ of $\text{Aut} \ X$ is dihedral provided there exist two vertices $u$ and $v$ belonging to distinct orbits of $\rho$ and $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^\#$ such that $B$ coincides with the union $\langle \rho^m \rangle u \cup \langle \rho^m \rangle v$. Our first lemma states that all blocks are of one of these two kinds.

**Lemma 7.1.** Let $n \geq 3$, $X \neq K_{2n}$, be a dihedrant of order $2n$, and let $B$ be an imprimitivity block system of $\text{Aut} \ X$. Then the blocks in $B$ are either all cyclic or all dihedral.

**Proof.** The proof is straightforward. Just take an arbitrary vertex $u \in B$ and let $m$ be the smallest positive integer with the property that $\rho^m u \in B$. Then either $B = \langle \rho^m \rangle u$ or there exists $v \in B \setminus \langle \rho^m \rangle u$. In the latter case $B = \langle \rho^m \rangle u \cup \langle \rho^m \rangle v$. ■

**Lemma 7.2.** Let $n \geq 3$, $X \neq K_{2n}$, be a 2-arc-transitive dihedrant of order $2n$, let $B$ be an imprimitivity block system of $A = \text{Aut} \ X$, let $D = \langle \rho, \tau \rangle$ be a regular dihedral subgroup of $A$, and let $K$ be the kernel of the action of $A$ on $X$. Then the quotient group $A/K$ acts 2-arc-transitively on $X_B$, and moreover, the following statements hold.

(i) if the blocks in $B$ are cyclic then $X_B$ is a dihedrant admitting a regular dihedral action of the group $D/K$;

(ii) if the blocks in $B$ are dihedral then $X_B$ is a circulant admitting a regular cyclic action of the group $\langle \rho \rangle/(K \cap \langle \rho \rangle)$.

**Proof.** The fact that $\bar{A} = A/K$ acts 2-arc-transitively is immediate. Next, to prove (i) observe that if the blocks in $B$ are cyclic, then there exists $m \in \mathbb{Z}_n^\#$ such that the kernel $K$ coincides with the subgroup $\langle \rho^m \rangle$. Hence $D/K = D/K = D/D \cap K$ is a regular dihedral subgroup of $\bar{A}$, and so $X_B$ is a 2-arc-transitive dihedrant.

As for (ii), observe that if the blocks are dihedral, then there exists $m \in \mathbb{Z}_n \setminus \mathbb{Z}_n^\#$ such that the kernel $K$ strictly contains the subgroup $\langle \rho^m \rangle$. Moreover, we have that $\langle \rho \rangle/K \cap \langle \rho \rangle = \langle \rho \rangle/\langle \rho^m \rangle$ is a regular cyclic subgroup of $\bar{A}$ and so $X_B$ is a 2-arc-transitive circulant ■

**Lemma 7.3.** Let $n \geq 3$, $X \neq K_{2n}, C_{2n}$ be a 2-arc-transitive dihedrant of order $2n$ and let $B$ be an imprimitivity block system of $\text{Aut} \ X$ with blocks of length $k \geq 2$. If $|B| > 2$, then $X$ is a $k$-fold cover of the quotient $X_B$. In particular, if the blocks in $B$ are cyclic then $X$ is a regular $\mathbb{Z}_k$-cover of $X_B$.
Proof. The proof is immediate if $X$ is a cycle. So assume that $X \not\cong C_{2n}$. Clearly, because of 2-arc-transitivity, we have that for any two blocks $B, B' \in \mathcal{B}$, a vertex in $B$ has at most one neighbour in $B'$. The result will be proved if we show that either there are no edges with one end-vertex in $B$ and the other end-vertex in $B'$ or the bigraph $X[B, B']$ is isomorphic to $kkK_2$. Let $K$ denote the kernel of the action of $A = \text{Aut} X$ on $\mathcal{B}$. If the blocks in $\mathcal{B}$ are cyclic, the result is immediate by Lemma 7.1. Moreover, in this case $X$ is a regular $\mathbb{Z}_k$-cover of $X_\mathcal{B}$ for there exists $m \in \mathbb{Z}_k$ such that the kernel $K$ coincides with a nontrivial subgroup $\langle \rho^m \rangle$ of $\langle \rho \rangle$ of order $k$. We may therefore assume that the blocks in $\mathcal{B}$ are dihedral and that, with no loss of generality, there is a block $B \in \mathcal{B}$ containing vertices $u_0$ and $v_0$. Then $B$ coincides with the set $\langle \rho^m \rangle u \cup \langle \rho^m \rangle v = \{u_0, u_m, \ldots, u_{(k-1)m}, v_0, v_m, \ldots, v_{(k-1)m} \}$. Let $[S, T]$, where $S \subseteq \mathbb{Z}_n^*$ and $T \subseteq \mathbb{Z}_n$, be the corresponding symbol of $X$ relative to the triple $(\rho, u_0, v_0)$, and let $B_i = \rho^i B$ for each $i \in \mathbb{Z}_n$. If $S \neq \emptyset$, then the bigraphs $X[B_i, B_j]$ are clearly of the desired form. So assume that $S = \emptyset$. By Lemma 7.2, the quotient $X_\mathcal{B}$ is a 2-arc-transitive circulant (of valency at least 3) and is therefore, by [1, Theorem 1.1], isomorphic to one of $K_m$, $K_{m/2, m/2}$, or $K_{m/2, m/2} - m/2K_2$, where $m = n/k$. Suppose first that $X_\mathcal{B} \cong K_{m/2, m/2}$ or $X_\mathcal{B} \cong K_{m/2, m/2} - m/2K_2$. The symbol of the circulant $X_\mathcal{B}$ is therefore either $\mathbb{O}_m$ in the first case or $\mathbb{O}_m \setminus \{m/2\}$ with $m/2$ odd in the second case. But then it easily follows that $T \subseteq \mathbb{O}_n$ and hence $\langle T - T \rangle \subseteq \mathcal{E}_n$. This implies that $X$ is disconnected, a contradiction. We may therefore assume that $X_\mathcal{B} \cong K_m$. Consider two neighbours $v_i$ and $v_j$ of $u_0$ and the corresponding three blocks $B_0$, $B_i$, $B_j$. By 2-arc-transitivity there exists $\alpha \in A$ which fixes $u_0$ and interchanges $v_i$ and $v_j$. By the nature of the blocks, precisely one of $v_i$ and $v_j$ has a neighbour in $B_j$ and $B_i$, respectively. Say $v_i$ has a neighbour $u_{j+l} \in B_j$ for some $l \in \mathbb{Z}_k$. But then $\alpha$ takes the edge $v_iu_{j+l}$ into a non-edge, a contradiction. Hence $X$ is a $k$-fold cover of $X_\mathcal{B}$. This completes the proof of Lemma 7.3. 

We now turn our attention to special kinds of imprimitivity block system, namely those with blocks of length 2 or 4.

Lemma 7.4. Let $n \geq 3$, let $X = X(2n, S, T) \not\cong K_{2n}$, $C_{2n}$ be a connected 2-arc-transitive dihedral of order $2n$, let $U$ and $V$ be the two orbits of the automorphism $\rho$ mapping according to the rule (1), and let $B$ be an imprimitivity block system of $\text{Aut} X$ with blocks of length 2 such that, for each $B \in \mathcal{B}$, we have $|B \cap U| = 1$ and $|B \cap V| = 1$. Then $X_\mathcal{B} \cong K_n$ and $X \cong K_{n, n} - nK_2$.

Proof. Clearly, modulo a possible relabeling of the vertices of $X$, we may assume that the blocks in $\mathcal{B}$ have the form $\{u_i, v_i\}$, $i \in \mathbb{Z}_n$. By Lemma 7.2, it follows that $X_\mathcal{B}$ is a 2-arc-transitive circulant, and hence, in view of [1, Theorem 1.1], isomorphic to one of $K_n$, $K_{n/2, n/2}$ or $K_{n/2, n/2} - n/2K_2$. Also,
Lemma 7.3 implies that $X$ is a 2-fold cover of $X_B$, and as such a regular $\mathbb{Z}_2$-cover with the permutation $\omega$ mapping according to the rule

$$\omega u_i = v_i, \ \omega v_i = u_i, \ i \in \mathbb{Z}_n,$$  \hspace{1cm} (18)

as the generator of the kernel $K$ of the action of $A = \text{Aut} X$ on $B$. But $\omega$ commutes with $\rho$ and so $X$ is also a Cayley graph of the abelian group $\langle \rho, \omega \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$. If we identify the trivial element of this group with the vertex $u_0$, then the three involutions $\omega, \rho^{n/2}$ and $\omega \rho^{n/2}$ are identified, respectively, with vertices $u_0, u_{n/2}$ and $v_{n/2}$. Furthermore $T = -T$ is a symmetric subset of $\mathbb{Z}_n$ and $S \cap T = \emptyset$.

If $X_B \cong K_n$ then, by [20, Theorem 1.1], $X$ is necessarily the canonical double cover of $X_B$ and thus isomorphic to $K_{n,n} - nK_2$. In what follows we may therefore assume that $X_B$ is isomorphic either to $K_{n/2,n/2}$ or to $K_{n/2,n/2} - n/2K_2$. We derive a contradiction in both cases.

**Case 1.** $X_B \cong K_{n/2,n/2} - n/2K_2$.

In particular, since $X_B$ is a circulant, we have that $n/2$ is odd and $S \cup T = \mathbb{O}_n \setminus \{n/2\}$. In view of the conditions on $S$ and $T$ we have that none of vertices $v_0$, $u_{n/2}$ and $v_{n/2}$ is in $N(u_0)$. Next, note that $N^2(u_0) = \{u_x : x \in (S + S) \cup (T + T)\} \cup \{v_y : y \in S + T\}$, and since, by assumption, $n/2$ is odd we have that $n/2 \notin (S + S) \cup (T + T) \cup (S + T)$. It follows that none of these vertices is in the set $N^2(u_0) \setminus N^2(u_0)$. But these three vertices correspond to the three involutions in $\langle \rho, \omega \rangle$, and so the above fact is a clear contradiction with Theorem 6.9.

**Case 2.** $X_B \cong K_{n/2,n/2}$.

Note that $S \cap T = \emptyset$ and that $S \cup T = \mathbb{O}_n$. First, we show that

$$S + T = \mathbb{E}_n^\#,$$  \hspace{1cm} (19)

Assume the contrary. Then for some $2j \in \mathbb{Z}_n^\#$ we have $(S + 2j) \cap T = \emptyset$. Consequenctly, $S + 2j \subseteq S$, and similarly $T + 2j \subseteq T$. In fact, $S + 2j = S$ and $T + 2j = T$. Consider the vertices $u_0$ and $u_{2j}$. It follows that $|N(u_0) \cap N(u_{2j})| = |S| + |T| = n/2$, the valency of $X$. But then the only possibility is that $X \cong 2K_{n/2,n/2}$, contradicting connectedness of $X$, and thus proving (19).

We now consider the condition on the number $\lambda$ of common neighbours of two vertices at distance 2, say $u_0$ in $U$ and the other, say $v_x$, in $V$. We have that $\lambda = |N(u_0) \cap N(v_x)| = |S \cap T + x| + |T \cap S + x| = 2|S \cap T + x|$. Then, by (19),

$$|S \cap (T + x)| = \begin{cases} \lambda/2, & x \in \mathbb{E}_n^\# \\ 0, & x \in \mathbb{O}_n \cup \{0\}. \end{cases}$$

We may translate this into the functional equation

$$\chi_S \ast \chi_T = \lambda \chi_{\mathbb{E}_n^\#}$$  \hspace{1cm} (20)

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where $\chi_M$ denotes the characteristic function of a subset $M$ of $\mathbb{Z}_n$ and $\ast$ denotes the convolution of functions. Let $i$ be the imaginary unit and $\xi = e^{2\pi i / n}$ be the $n$th root of unity. Applying the discrete Fourier transform (which converts convolutions to products) to (20) we get, for each $j \in \mathbb{Z}_n$,

$$2\sum_{k \in S} (\xi^j)^\ast \sum_{l \in T} (\xi^j)^l = \lambda((\xi^j)^2 + (\xi^j)^4 + \cdots + (\xi^j)^{n-2}).$$

(21)

Note that if $n/2$ is odd, then precisely one of the two vertices $u_{n/2}$ and $v_{n/2}$ is a neighbour of $u_0$. Besides, the third vertex $v_0$ is at distance at least 3 from $u_0$. Hence $N^2(u_0)$ contains at most one of these three vertices (corresponding to involutions in $\langle \rho, \omega \rangle$). It can be seen that $FQ_5$ is not a dihedral and so Theorem 6.9 implies that $X \cong K_{n,n} - nK_2$ is the only possibility. A contradiction, as the latter is clearly not a 2-fold cover of $K_{n/2,n/2}$.

We may therefore assume that $n/2$ is even. Plugging into (21) the value $j = n/4$, we get 0 on the left-hand side of the equation. Namely, as $n$ is divisible by 4, we have with each $i = \xi^{4k+1}$ also $-i = \xi^{n-4k-1}$ inside each of the two sums. But the right-hand side of this equation is nonzero; in fact it equals $-\lambda$.

These contradictions show that neither this case can occur, completing the proof of Lemma 7.4.

The next lemma deals with blocks of length 4.

**Lemma 7.5.** Let $n \geq 3$, let $X = X(2n, S, T) \cong K_{2n}, C_{2n}$ be a connected 2-arc-transitive dihedral of order $2n$, let $U$ and $V$ be the two orbits of the automorphism $\rho$ mapping according to the rule (1), let $B$ be an imprimitivity block system of $A = \text{Aut} X$, and let $K$ be the kernel of the action of $A$ on $B$.

If $|B| \geq 2$ and if the blocks in $B$ are of length 4 and are such that $|B \cap U| = 2$ and $|B \cap V| = 2$ for each $B \in B$, and if $K$ acts transitively on each $B \in B$, then the following statements hold.

(i) $n = 2(q + 1)$, where $q$ is an odd prime power, $X_B \cong K_{q+1}$ and $X \cong K_{q+1}^4$; and

(ii) $X$ has a symbol of the form $[\emptyset, T_1 \cup T_2]$, where $T_1 \cap T_2 = \emptyset$, $T_1 = -T_1$, $T_2 = -T_2 + q + 1$ and $\{[T_1], [T_2]\} = \{(q-1)/2, (q+1)/2\}$.

**Proof.** Since $|B| \geq 2$ and $|B| = 4$ for $B \in B$ and since $K$ is transitive on each $B \in B$, we have that $X$ is a regular 4-fold cover of $X_B$. With no loss in generality we can assume that $B = \{u_0, u_{n/2}, v_0, v_{n/2}\}$ is a block in $B$. Among others this implies that

$$0, n/2 \notin S \cup T$$

(22)

and that $S \cap (S + n/2) = \emptyset$ and

$$T \cap (T + n/2) = \emptyset.$$  

(23)
For each \( i \in \mathbb{Z}_n \) let \( B_i = \rho^i B \). Since the blocks in \( B \) are dihedral, Lemma 7.2 implies that the quotient \( X_B \) is a 2-arc-transitive circulant (of order \( n/2 \)) and thus, in view of [1, Theorem 1.1], isomorphic to one of \( K_{n/2}, K_{n/4,n/4} \) or \( K_{n/4,n/4} - (n/4)K_2 \). In particular, the valency \( \text{val} X \) of \( X \) attains one of the following values:

\[
\text{val} X \in \{n/4 - 1, n/8, n/8 - 1\} \tag{24}
\]

There exists an automorphism \( \sigma \) of \( X \) belonging to the kernel \( K \) such that the restriction \( \sigma^B \) is either \((u_0v_0)(u_n/2v_n/2)\) or \((u_0^\rho v_0^\rho)(u_n/2v_n/2)\). We split the argument accordingly.

**Case 1:** \( \sigma^B = (u_0v_0)(u_n/2v_n/2) \).

If \( \sigma \) interchanges \( u_i \) with \( v_i \) for each \( i \in \mathbb{Z}_n \), then \( X \) is a Cayley graph of the abelian group \( \langle \rho, \sigma \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2 \), which has precisely three involutions. If we identify the trivial element of this group with the vertex \( u_0 \), then the three involutions \( \sigma, \rho^i, \sigma \rho^j \) are identified, respectively, with vertices \( v_0, u_n/2v_n/2 \). But all of these vertices are in the same block as \( u_0 \) and hence at the same distance from \( u_0 \). In view of (22) this distance is at least 3, contradicting Theorem 6.9.

We may therefore assume that there exists a decomposition \( \{L, M\} \) of \( \mathbb{Z}_n \) such that \( 0 \in L \) and such that \( K = \langle \rho^n/2, \sigma \rangle \), where \( \sigma \) maps according to the following rule.

\[
\sigma^{B_i} = \begin{cases} 
(u_i v_i)(u_{i+n/2} v_{i+n/2}), & i \in L \\
(u_{i+n/2} v_{i+n/2})(u_i v_i), & i \in M.
\end{cases}
\]

Clearly, \( L = L + n/2 \) and \( M = M + n/2 \). Choose any two adjacent blocks \( B_i \) and \( B_j \) in \( B \). Then we may easily see that there exists a decomposition \( \{T_1, T_2\} \) of \( T \) such that \( T_1 = -T_1 \) and \( T_2 = -T_2 + n/2 \) and that, moreover,

\[
 j - i \in S \cup T_1 \iff i, j \in L \text{ or } i, j \in M
\]

and

\[
 j - i \in T_2 \iff i, j \in M \text{ or } i, j \in L.
\]

Consequently,

\[
 S \cup T_1 \subseteq L \quad \text{and} \quad T_2 \subseteq M. \tag{25}
\]

Set \( L_0 = L \cap E_n, L_1 = L \cap \Theta_n, M_0 = M \cap E_n \) and \( M_0 = M \cap \Theta_n \). Furthermore, let \( \mathcal{L}_0 = \{B_i : i \in L_0\}, \mathcal{L}_1 = \{B_i : i \in L_1\}, \mathcal{M}_0 = \{B_i : i \in M_0\} \) and \( \mathcal{M}_1 = \{B_i : i \in M_1\} \).

We have two possibilities.

**Subcase 1.1:** \( X_B \not\cong K_{n/2} \).

In this case \( X_B \) is isomorphic either to \( K_{n/4,n/4} \) or to \( K_{n/4,n/4} - n/4K_2 \). Consequently, the symbol of the circulant \( X_B \) is either \( \Theta_n \) in the first case or \( \Theta_n \setminus \{n/4\} \), and moreover \( n/4 \) odd, in the second case. This means that the bigraphs \( [\mathcal{L}_0, \mathcal{L}_1] \)

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and $[M_0, M_1]$ which are $|S \cup T_1|$-regular, and the bigraphs $[L_0, M_1]$ and $[M_0, L_1]$, which are $|T_2|$-regular graphs, exhaust all the edges of $X_B$. In particular, neither the subgraph generated by $S \cup T_1$-edges nor the subgraph generated by $T_2$-edges is connected. But this is impossible as $Z^{n/2}_{n/2} \subseteq \bigcup_{n/2}$ and each unit in the symbol of the circulant $X_B$ gives rise to a connected subgraph of $X_B$. Consequently, some of the sets $L_0, L_1, M_0, M_1$ must be empty. But by assumption $0 \in L$ and so $L_0 \neq \emptyset$. This leaves us with one possibility only: $L = E_n$ and $M = \bigcup_{n}$. It follows that $S \cup T_1 = \emptyset$ and so $T = T_2$. Hence a symbol of $X$ has the form $[\emptyset, T]$ with $T \subseteq \bigcup_{n}$. But then $\langle T - T \rangle \subseteq E_n$, and so $X$ is disconnected, a contradiction.

Subcase 1.2: $X_B \cong K_{n/2}$.

Since any two $B_i, B_j, i \neq j$, are adjacent, (25) gives us

$$(S \cup T_1) \cup (S \cup T_1 + n/2) = L^\# \setminus \{n/2\} \text{ and } T_2 \cup (T_2 + n/2) = M. \quad (26)$$

Besides, $L + L \subseteq L$, $M + M \subseteq L$ and $L + M \subseteq M$. Since both $L$ and $M$ are non-empty subsets of $\mathbb{Z}_n$, it follows that $L$ is a subgroup of $\mathbb{Z}_n$ of index 2. In other words, $L = \bigcup_{n}$, and $M = \bigcup_{n}$. Note that in view of [20, Theorem 1.1], we have $n = 2(q + 1)$, where $q \equiv 1 \pmod{4}$ is a prime power, and $X \cong K_{q+1}^{4}$. But then, by Theorem 5.3, we may assume that $S = \emptyset$. Applying (26) we have $T_1 \subseteq \bigcup_{n}^{\#}$, $T_2 \subseteq \bigcup_{n}$ and so, in view of (23), their cardinalities $|T_1|$ and $|T_2|$ are respectively $(q - 1)/2$ and $(q + 1)/2$.

Case 2: $\sigma^B = (u_0v_0/r_{i+n/2}^{p_{i+n/2}})$.

If the restriction $\sigma^B$ is $(u_0v_0/r_{i+n/2}^{p_{i+n/2}})$ for each $i \in \mathbb{Z}_n$, then it easily follows that $X$ is a Cayley graph of the abelian group $(\rho, \sigma) \cong \mathbb{Z}_n \times \mathbb{Z}_2$, which has precisely three involutions. These involutions may be identified with vertices $u_{n/2}$, $v_{n/4}$ and $v_{3n/4}$, provided the trivial element is identified with $u_0$. Clearly, $u_{n/2}$ is at distance at least 3 from $u_0$. We may then apply Theorem 6.9 to deduce that $X$ is either isomorphic to the complete bipartite graph with a 1-factor removed (the three involutions at distance 1, 2 and 3 from $u_0$) or is isomorphic to some 2-fold cover of a complete bipartite graph (with two involutions at distance 2 and one at distance 4 from $u_0$). Consequently, $\text{val}X$ is either $n - 1$ or $n/2$, contradicting (24).

The rest of the argument is almost identical to the one used in Case 1. Namely, we can assume that there exists a decomposition $\{L, M\}$ of $\mathbb{Z}_n$ such that $0 \in L$ and that $K = \langle \rho^{n/2}, \sigma \rangle$, where $\sigma$ maps according to the following rule.

$$\sigma^B = \begin{cases} 
(u_0v_iu_{i+n/2}v_{i+n/2}), & i \in L \\
(u_0v_{i+n/2}^iv_{i+n/2}^i), & i \in M.
\end{cases}$$

We have $L = L + n/2$ and $M = M + n/2$. Choosing any two adjacent blocks $B_i$ and $B_j$ in $B$, we see that there exists a decomposition $\{T_1, T_2\}$ of $T$ such that $T_1 = -T_1$ and $T_2 = -T_2 + n/2$ and that, moreover,

$$j - i \in S \cup T_2 \iff i, j \in L \text{ or } i, j \in M$$
and

\[ j - i \in T_1 \iff i \in L, j \in M \text{ or } i \in M, j \in L. \]

Consequently,

\[ S \cup T_2 \subseteq L \text{ and } T_1 \subseteq M. \tag{27} \]

Comparing (27) with (25) we see that the only distinction between Cases 1 and 2 is that the roles of \( T_1 \) and \( T_2 \) are interchanged. As in Subcase 1.1 we get that \( X_G \not\cong K_{n/2} \) forces \( X \) to have a symbol of the form \([0, T]\) with \( T \subseteq \Omega_n \). Hence \( \langle T - T' \rangle \subseteq E_n \), and so \( X \) is disconnected, a contradiction. On the other hand, if \( X_G \cong K_{n/2} \) then, by [20, Theorem 1.1], we have \( n = 2(q + 1) \), where \( q \equiv 3 \pmod{4} \), and \( X \cong K_{q+1} \). Moreover, \( S = \emptyset \), \( T_1 \subseteq \Omega_n \), and \( T_2 \subseteq \Omega_{n^2} \), and in view of (23), the cardinalities \( |T_1| \) and \( |T_2| \) are respectively \((q + 1)/2\) and \((q - 1)/2\). □

Let us mention that if \((T_1, T_2)\) is a pair of subsets satisfying part (ii) of Lemma 7.5, then so are the pairs \((T_1 + x, T_2 + x)\), for \( x = (q + 1)/2, q + 1, 3(q + 1)/2 \). In particular, this means that the cardinalities of the sets \( T_1 \) and \( T_2 \) may take both values \((q - 1)/2\) and \((q + 1)/2\). Namely, if say \(|T_1| = (q + 1)/2\) and \(|T_2| = (q - 1)/2\), then by letting \( T_1' = T_2 + (q + 1)/2 \) and \( T_2' = T_1 + (q + 1)/2 \), we have that \( T_1' = -T_1, T_2' = -T_2 + q + 1 \), and \(|T_1'| = (q - 1)/2\), \(|T_2'| = (q + 1)/2\). Now, it can be easily seen that precisely one of the above mentioned four pairs of subsets has the property that the first component, that is the symmetric one, contains 0. We call this pair the canonical form of the corresponding symbol of the graph \( K_{q+1} \).

We summarize the above discussion in the following corollary of Lemma 7.5.

**Corollary 7.6.** Let \( q \) be an odd prime power. Then the 2-arc-transitive dihedral \( K_{q+1}^4 \) has a symbol of the form \([0, T_1 \cup T_2]\), where \( T_1 \subseteq \Omega_{2(q+1)} \setminus \{q + 1\}, T_2 \subseteq \Omega_{2(q+1)} \), \( 0 \in T_1, T_1 = -T_1, \) and \( T_2 = -T_2 + q + 1 \). Moreover,

(i) if \( q \equiv 1 \pmod{4} \) then \(|T_1| = (q + 1)/2\) and \(|T_2| = (q - 1)/2\); and

(ii) if \( q \equiv 3 \pmod{4} \) then \(|T_1| = (q - 1)/2\) and \(|T_2| = (q + 1)/2\).

Table 1 below gives the canonical forms \((T_1, T_2)\) of symbols \([0, T_1 \cup T_2]\) of the graphs \( K_{q+1}^4 \) for all prime powers \( q \leq 25 \). The computations have been carried out using MAGMA [4]. See also Figure 1 where the graphs \( K_{q+1}^4 \), \( q = 3, 5, 7, 9 \) are shown.

**TABLE 1:** The sets \( T_1 \) and \( T_2 \).

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<table>
<thead>
<tr>
<th>$q$</th>
<th>$n$</th>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>$[0]$</td>
<td>$[1,3]$</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>$[0,2,10]$</td>
<td>$[7,11]$</td>
</tr>
<tr>
<td>7</td>
<td>16</td>
<td>$[0,2,14]$</td>
<td>$[3,5,9,15]$</td>
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<td>9</td>
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</tr>
<tr>
<td>13</td>
<td>28</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>21</td>
<td>48</td>
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</tr>
<tr>
<td>23</td>
<td>52</td>
<td>$[0,6,14,16,18,22,24,26,30,34,36,38,46]$</td>
<td>$[9,17,27,29,31,33,37,41,45,47,49,51]$</td>
</tr>
</tbody>
</table>

We wrap up this section with two propositions. The first one gives a description of all possible symbols of the complete bipartite graphs and the complete bipartite graphs with a 1-factor removed. The second one suggests a uniqueness of these two classes of graphs among 2-arc-transitive dihedrants with symbols of the form $[0, T]$.

**Proposition 7.7.** Let $n \geq 3$ and let $X$ be a dihedrant of order $2n$ with symbol $[S, T]$ such that $S \neq \emptyset$ and $0 \in T$. Then the following statements hold.

(i) If $X \cong K_{n,n}$, then $n$ is even, and $[S, T] = [\emptyset_n, E_n]$.

(ii) If $X \cong K_{n,n} - nK_2$, then $n$ is even and either $[S, T] = [\emptyset_n, E_n \setminus \{2k\}]$ for some $2k \neq 0$, or $n/2$ is odd and $[S, T] = [\emptyset_n \setminus \{n/2\}, E_n]$.

**Proof.** That $n$ is even in both cases is clear. Since both $K_{n,n}$ and $K_{n,n} - nK_2$ are bipartite graphs, and hence triangle-free graphs, we have

$$S \cap (S + S) = \emptyset \quad \text{and} \quad S \cap (T + T) = \emptyset.$$  \hfill (28)

In particular, since $0 \in T$, we have

$$S \cap T = \emptyset.$$  \hfill (29)

The proof of (i) is now at hand. Since the valency of $K_{n,n}$ is $n = |S| + |T| = |S \cup T|$, it follows by (29), that $S \cup T = \mathbb{Z}_n$. In particular, $T = -T$ and therefore, by (28), $S \cap 2(S \cup T) = \emptyset$ and so $S \cap E_n = \emptyset$. Hence $S \subseteq \emptyset_n$ and so $E_n \subseteq T$. Now assume that
there exists $t = 2i + 1 \in T$ and take $s = 2j + 1 \in S$. Then $s - t = 2(j - i) \in \mathbb{E}_n \subseteq T$. So $s \in T + t = \neg T + t$, contradicting (28). We conclude that $S = \mathbb{O}_n$ and $T = \mathbb{E}_n$.

As for part (ii), suppose first that $\mathbb{E}_n \not\subseteq (S + S) \cup (T + T)$. Let, in view of (29), $x$ be the sole element of $\mathbb{Z}_n \setminus S \cup T$. Then either $x = -x = n/2$ or $-x \in T$. The first case cannot occur for it implies $T = \neg T$ which clearly forces $\mathbb{E}_n \subseteq (S + S) \cup (T + T)$. As for the second case it implies that $T \setminus \{x\}$ is a symmetric subset of $\mathbb{Z}_n$, forcing $\mathbb{E}_n \setminus \{2x\} \subseteq (S + S) \cup (T - T)$. We may therefore assume that $2x \notin (S + S) \cup (T - T)$. Then $S \cap (S + 2x) = \emptyset$ as well as $T \cap (T + 2x) = \emptyset$. It follows that $u_{2x} = u_{-2x}$, is the vertex at distance 3 from $u_0$. Hence $4x = 0$ and so $x \in \{n/2, 3n/2\}$. In particular, $n/2$ is even and moreover $\mathbb{E}_n \setminus \{n/2\} \subseteq (S + S) \cup (T - T)$. But then, by (28), $S \subseteq \mathbb{O}_n \cup \{n/2\}$ and so $\mathbb{E}_n \setminus \{x, n/2\} \subseteq T$. But $T \cap (T + n/2) = \emptyset$ and so $\mathbb{E}_n \setminus \{x, n/2\} \cap (\mathbb{E}_n \setminus \{x, n/2\} + n/2) = \emptyset$. It follows that $n/2 - 2 \leq n/4$, forcing $n = 8$. With no loss of generality $x = n/4 = 2$ and so this gives us the following four possibilities for the sets $S$ and $T$: either $S = \{4\}$ and $T = \{0, 1, 3, 5, 6, 7\}$, or $S = \{1, 4, 7\}$ and $T = \{0, 3, 5, 6\}$, or $S = \{3, 4, 5\}$ and $T = \{0, 1, 6, 7\}$, or $S = \{1, 3, 4, 5, 7\}$ and $T = \{0, 6\}$. But in all of these cases there is a triangle in $X$, contradicting (28).

We may therefore assume that $\mathbb{E}_n \subseteq (S + S) \cup (T + T)$. Then, by (28), we have $S \subseteq \mathbb{O}_n$. Clearly, if $S = \mathbb{O}_n$ then $T = \mathbb{E}_n \setminus \{2k\}$ for some $2k \neq 0$. So assume that $S \neq \mathbb{O}_n$ and take $z \in \mathbb{O}_n \setminus S$. Assume first that $z \in T$ and consider the sums $s + z$ and $-s + z$ for some $s \in S$. Both $s + z$ and $-s + z$ are even, and so if $s \neq -s$ then one of the above two sums is in $T$, contradicting (28). So $S \subseteq \{n/2\}$, as the only remaining possibility. But then, since at most one of $1, -1, 1 + n/2, -1 + n/2$ does not belong to $T$, we have again a contradiction with (28). All of these show that $z \notin T$ and so $T \subseteq \mathbb{E}_n$. Therefore $S = \mathbb{O}_n \setminus \{z\}$ and $T = \mathbb{E}_n$. Since $S = -S$, it follows that $z = -z$ and so $z = n/2$. In particular, $n/2$ is odd. Hence proof.

Lemma 7.8. Let $n \geq 3$ and let $X \not\subseteq C_{2n}$ be a connected 2-arc-transitive Cayley graph of a dihedral group of order $2n$ having a symbol $[\emptyset, T]$, where $T = -T$ is a symmetric subset of $\mathbb{Z}_n$. Then $X$ is isomorphic either to $K_{n,n}$ or to $K_{n,n} - nK_2$.

Proof. Since $T = -T$, it follows that $X$ is also a Cayley graph of the abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$. More precisely, the permutation $\omega$, mapping according to the rule (18) together with $\rho$ generates a subgroup of $\text{Aut} \ X$ isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_2$. As in the proofs of Lemmas 7.4 and 7.5, we identify the trivial element of the group and the three involutions $\omega, \rho^{n/2}$ and $\omega \rho^{n/2}$, respectively, with vertices $v_0, v_0, u_{n/2}$, and $v_{n/2}$. We then apply Theorem 6.9 to deduce that $X$ is isomorphic either to $K_{n,n}$ or to $K_{n,n} - nK_2$. ■

8 Proving Theorem 2.1

The following group-theoretic results will be needed in the proof of Theorem 2.1,
or rather in the proof of its modified version Theorem 8.4.

**Proposition 8.1.** (Burnside, [42, Theorem 7.3]) A transitive permutation group of prime degree \( p \) is either doubly transitive or solvable, in which case it may be identified with the group of all affine transformations \( f_{a,b} : x \to ax + b \ (a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p) \) of the field \( \mathbb{Z}_p \).

**Proposition 8.2.** (Schur, [53, Theorem 25.3]) A primitive group containing a regular cyclic subgroup of composite order is doubly transitive.

**Proposition 8.3.** (Wielandt, [53, Theorem 25.6]) A primitive group containing a regular dihedral subgroup is doubly transitive.

We now restate Theorem 2.1 in a slightly modified form. Let \( G_1 \) denote the class of graphs containing: \( C_{2n}, n \geq 3 \) and \( K_{2n}, n \geq 3, n \geq 3 \) and let \( G_2 \) be the class of graphs containing: \( K_{n,m}, n \geq 3, K_{n,n} - nK_2, n \geq 3, K_{q+1, q}^{q}, q \) an odd prime power, \( B(H_1), B'(H_1) \), and \( B(PG(d,q)) \) and \( B'(PG(d,q)) \), with \( d \geq 2 \) and \( q \) a prime power.

**Theorem 8.4.** Let \( n \geq 3 \), let \( X \) be a connected, 2-arc-transitive Cayley graph of a dihedral group \( D = \langle \rho, \tau \rangle \) of order \( 2n \), where \( \rho \) and \( \tau \) map according to the respective rules (1) and (2). Then one of the following occurs.

(i) either \( X \in G_1 \cup G_2 \); or

(ii) \( X \) is a regular cyclic cover of a graph in \( G_2 \); more precisely: there exists a proper divisor \( m \) of \( n \) such that the set \( B \) of orbits of \( \langle \rho^m \rangle \) is an imprimitivity block system of \( AutX \) relative to which \( X \) is a regular \( \mathbb{Z}_{n,m} \)-cover of \( X_B \), the latter being a graph in \( G_2 \) admitting a regular dihedral group \( D / \langle \rho^m \rangle \).

**Proof.** Assume that, by contradiction, \( X \) is a minimal counterexample to the statement of Theorem 8.4. Since \( X \not\cong C_{2n} \), we have that

\[
valX \geq 3. \tag{30}
\]

Next, since \( X \not\cong K_{2n} \), the girth of \( X \) is at least 4, and moreover, Proposition 8.3 implies that \( A = AutX \) is imprimitive. Choose a minimal block of \( A \), say of cardinality \( k \geq 2 \), and let \( B \) be the corresponding imprimitivity block system of \( A \). Let \( K \) denote the kernel of the action of \( A \) on \( B \) and let \( \bar{A} = A/K \) denote the corresponding quotient group. Clearly, in view of the minimality of \( B \), the restriction \( A_B^B \) of the setwise stabilizer \( A_B \) to \( B \) is primitive. Now \( B \) is also an imprimitivity block system for \( D \). Besides, \( D_B^B \) is a regular (cyclic or dihedral) subgroup of a primitive group \( A_B^B \). Hence Propositions 8.2 and 8.3 together imply that either

\[
A_B^B \text{ is doubly transitive or } k = p \text{ is prime.} \tag{31}
\]

We distinguish two different cases.
Case 1. $|\mathcal{B}| = 2$.

Then $\mathcal{B} = \{B, B'\}$ and we have that the kernel $K$ coincides with the setwise stabilizers $A_B = A_B'$. Apply (31) and Proposition 8.1 to have that

$$K_B^R \text{ and } K_B'^R \text{ are either solvable of prime degree or doubly transitive.} \quad (32)$$

Suppose first that $K$ acts unfaithfully on the two blocks $B$ and $B'$. Then we may assume $K_B^R \neq \text{id}$. But $K_B'^R$ is a normal subgroup of $K_B^R$, a primitive group, and so $K_B'^R$ is transitive by [53, Theorem 8.8]. This forces $X = X[B, B'] \cong K_{k,k}$ and so $X \in \mathcal{G}_2$.

Assume that $K$ acts faithfully on the two blocks $B$ and $B'$. We claim that in this case $K_B^R$ and $K_B'^R$ are doubly transitive. Note that, for $v \in B$, the set of neighbours $N(v)$ of $v$ must be an orbit of $K_v = A_v$. Moreover, 2-arc-transitivity of $X$ forces a doubly transitive action of $K_v$ on $N(v)$. So assume now that $k$ is a prime and that $K_B^R$ and $K_B'^R$ are solvable. Combining together Proposition 8.1 with the faithfulness of the action of $K$ on $B$ and $B'$ we see that $K_v$ is cyclic of order $d$, where $d$ divides $p - 1$. Therefore $K_v$ cannot be doubly transitive on $N(v)$ as $|N(v)| = v |X| \geq 3$ in view of (30). So (32) implies that $K_B^R$ and $K_B'^R$ are indeed doubly transitive.

If $K_B^R$ and $K_B'^R$ are permutationally equivalent then, for each $v \in B$, the vertex stabilizer $K_v$ has a fixed vertex $v' \in B'$ and two orbits $B \setminus \{v\}$ and $B' \setminus \{v'\}$. It follows that $N(v) = B' \setminus \{v'\}$ and so $X = |B, B'| \cong K_{k,k} - kK_2$ and $X \in \mathcal{G}_2$.

We may now assume that $K_B^R$ and $K_B'^R$ are permutationally inequivalent. Recall that $K_B^R$ and $K_B'^R$ are doubly transitive groups which contain a regular cyclic or dihedral group. Hence $X$ is isomorphic to the incidence/nonincidence graph of a symmetric design with a group of automorphisms acting doubly transitively on points and containing a regular cyclic or dihedral group. It may be deduced from [32, Theorem] that $X$ is isomorphic to one of the following graphs: the incidence and nonincidence graphs $B(PG(d,q))$ and $B'(PG(d,q))$, respectively, associated with the projective spaces $PG(d,q)$, $d \geq 2$, and the incidence and nonincidence graph $B(H_{11})$ and $B'(H_{11})$, respectively, of the unique Hadamard design $H_{11}$ on 11 points. Hence $X \in \mathcal{G}_2$.

All of the above contradict the choice of $X$ and show that this case cannot occur.

Case 2. $|\mathcal{B}| > 2$.

By (30), $X_B$ is not a cycle. Let us consider the setwise stabilizer $D_B$. It follows that either there exists $m \in \mathbb{Z}_n$ such that $D_B = \langle \rho^m \tau \rangle$, or there exists $m \in \mathbb{Z}_n^\# \setminus \mathbb{Z}_n^*$ such that either $D_B = \langle \rho^m \rangle$ or $D_B = \langle \rho^m, \tau \rangle$.

Suppose first that $D_B = \langle \rho^m \tau \rangle$. Then $X$ is a 2-fold cover of $X_B$. Also, a block in $\mathcal{B}$ contains one vertex from the orbit $U$ and one vertex from the orbit $V$. By Lemma 7.4 it follows that $X \cong K_{n,n} - nK_2$ and so $X$ belongs to $\mathcal{G}_2$.

Suppose next that $D_B = \langle \rho^m \rangle$. Note that $id \neq \langle \rho^m \rangle \neq \langle \rho \rangle$ and that $\langle \rho^m \rangle \leq K$. In particular, $K_B^R$ is a 2-extension of the cyclic group $\langle \rho^m \rangle^B$ and thus the minimality
of the block $B$ forces $|B| = 4$, $m = n/2$ and so $K \cong \mathbb{Z}_2^2$, for in all other cases the orbits of $\langle \rho^m \rangle$ would form an imprimitivity block system of a doubly transitive group $A_B^B$. Moreover, the blocks in $B$ are dihedral consisting of two vertices from each of the two orbits of $\rho$. By Lemma 7.5 we have that $n = 2(q + 1)$, and $X_B \cong K_{q+1}$ and $X \cong K_{2q+1}^4$, where $q \equiv 1 \pmod{4}$ is an odd prime power. Thus $X \in \mathcal{G}_2$.

Suppose finally that $D_B = \langle \rho^m \rangle$. As above $id \neq \langle \rho^m \rangle \neq \langle \rho \rangle$. Clearly, $D_B$ is a normal subgroup of $D$ and so $\langle \rho^m \rangle \leq K$. Of course, $D_B$ is regular on $B$ and so $id \neq K = \langle \rho^m \rangle$. Since $K$ is a cyclic normal subgroup of $A$, it follows that every subgroup of $K$ is normal in $A$, and thus the orbits of this subgroup form a block system of $A$. The minimality of blocks in $B$ then forces $k = p$, a prime and $K \cong \mathbb{Z}_p$. Thus $X$ is a regular $\mathbb{Z}_p$-cover of $X_B$. Moreover, since the blocks in $B$ are cyclic, Lemma 7.2 implies that $X_B$ is itself a 2-arc-transitive dihedral, of order $2m$, where $m = n/p < n$, admitting a dihedral regular subgroup $D/\langle \rho^m \rangle$, The choice of $X$ as a minimal counterexample now leaves us with the following three possibilities.

First, if $X_B$ belongs to $\mathcal{G}_1$ then it is isomorphic to $K_{2m}$. But then $X$ is a 2-arc-transitive regular $\mathbb{Z}_p$-cover of $K_{2m}$. Such graphs were classified in [20, Theorem 1.1], and it follows that $X \cong K_{2m,2m} - 2mK_2$. Hence $X \in \mathcal{G}_2$.

Second, if $X_B$ belongs to $\mathcal{G}_2$ then $X$ is a regular cyclic cover of a graph in $\mathcal{G}_2$.

Finally, suppose that $X_B$ is itself a regular cyclic cover of a graph $Y$ in $\mathcal{G}_2$. Then there exists a non-trivial proper subgroup of $\langle \rho \rangle/\langle \rho^m \rangle$ whose orbits form an imprimitivity block system of $\overline{A}$ relative to which $X_B$ is a regular cyclic cover of $Y$. It can be easily seen that, in view of Lemma 7.3, this implies the existence of a proper divisor $l$ of $m$ such that the set $C$ of orbits of $\langle \rho \rangle$ is an imprimitivity block system of $A$ relative to which $X$ is a regular $\mathbb{Z}_{n/l}$-cover of $X_B \cong Y$.

As in Case 1, all of the above contradict the choice of $X$ and show that neither this case can occur. This completes the proof of Theorem 8.4. ■

Proof of Theorem 2.1. Theorem 2.1 is an immediate consequence of Theorem 8.4.

References


Figure 1: The 2-arc-transitive dihedrants $K_{q+1}^4$, $q = 3, 5, 7, 9.$