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CARTESIAN PRODUCT
$K_{m,m,m} \times C_{2n}$

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The Genus of the Cartesian Product $K_{m,m,m} \times C_{2n}$

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Abstract

The genus of $K_{m,m,m} \times C_{2n}$ is determined for $m \geq 1$ and for all $n \geq 3$ and $n = 1$. For $n = 2$ both lower and upper bounds are given.

Let $\times$ denote the Cartesian product of graphs.

**Theorem 1.** The genus of $K_{m,m,m} \times C_{2n}$ for $m \geq 1$, $n \geq 3$ is given by the formula:

$$\gamma(K_{m,m,m} \times C_{2n}) = 1 + m(m - 1)n$$

**Proof.** For $m = 1$ we have $K_{1,1,1} = C_3$ and $C_3 \times C_{2n}$ is obviously toroidal. From here on let $m \geq 2$. We first prove $\gamma(K_{m,m,m} \times C_{2n}) \leq 1 + m(m - 1)n$. We start with 2n copies of triangulation of $K_{m,m,m}$ in a surface $S_g$ of genus $g = (m - 1)(m - 2)/2$. For $m = 3$ the surface $S_g$ is a torus as shown in Figure 1. In this particular case the embedding has 6 disjoint patchworks, two of which are indicated. In general there are 2m disjoint patchworks, two of which are needed in the construction. Since $C_{2n}$ is a bipartite 2-regular graph we may apply the patchwork method to embed $K_{m,m,m} \times C_{2n}$ into an orientable surface of genus $1 + m(m - 1)n$. For explanation of this classical method, see for instance \cite{3,4,5}. The two patchworks may be constructed for instance, by taking alternating edges of any Petrie walk of the well-known Hamilton embedding of $K_{n,n}$ in the surface of genus $(m - 1)(m - 2)/2$ and then augmenting the edges to appropriate triangles of $K_{n,n}$ in the same surface. We double-check the genus formula by the following argument.

1. There are 2n copies of $S_g$, arranged in a circle, each triangulated by a copy of $K_{m,m,m}$.

2. There are m tubes between any two consecutive $S_g$, giving the total number of tubes equal to $2mn$.

3. (2n - 1) tubes are needed to connect the 2n initial surfaces $S_0$ into a single surface $\Sigma_0$. Hence the final surface $\Sigma$ is homeomorphic to a sphere with $2mg + 2mn - (2n - 1) = 1 + m(m - 1)n$ handles attached. The embedding consists of $4m(m - 1)n$ triangles remaining in the original surfaces $S_g$ and $6mn$ quadrilaterals along the $2mn$ tubes. There are $2m + 2$ faces incident with any vertex: $2m - 2$ triangles and 4 quadrilaterals.

The proof that $\gamma(K_{m,m,m} \times C_{2n}) \geq 1 + m(m - 1)n$ follows.

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Let us take an embedding of a graph with vertices $x_1, x_2, \ldots, x_v$ and a total of $f$ faces. Let $f_k$ denote the total number of faces of size $k$ and let $a_k(x)$ denote the number of faces of size $k$ incident with a given vertex $x$. Clearly:

\[
val(x) = a_3(x) + a_4(x) + \ldots,
\]

\[
k f_k = a_k(x_1) + a_k(x_2) + \ldots + a_k(x_v),
\]

and

\[
f = f_3 + f_4 + \ldots
\]

For a vertex $x$ define its face contribution to be $\phi(x) = a_3(x)/3 + a_4(x)/4 + \ldots$. If a graph has $v$ vertices, $e$ edges then the genus of the graph is $\phi_0 = (\phi(x_1) + \phi(x_2) + \ldots + \phi(x_v))/v$ denote the average face contribution. Then $f = \phi(x_1) + \phi(x_2) + \ldots + \phi(x_v)$. This embedding can be expressed as: $\gamma = 1 + e/2 - v(1 + \phi_0)/2$. Therefore minimizing $\gamma$ is equivalent to maximizing $\phi_0$. In our case, $v = 6mn, e = 6n(m + 1)n$. Hence

\[
\gamma(K_{m,n} \times C_{2n}) \geq 1 + m(m - 1)n
\]

is equivalent to saying that for any embedding of $K_{m,n} \times C_{2n}$, we have $\phi_0 \leq (2m + 1)/3$. If we can show this inequality not only for the average face contribution but for the maximal face contribution we are done.

Let $t = a_3(x)$ be the number of triangles incident with a vertex $x$. Since $val(x) = 2m + 2$ it follows that $\phi(x) \leq (m + 1)/2 + t/12$. Since adjacent vertices in different copies of $K_{m,n}$ do not belong to a common triangle $0 \leq t \leq 2m$. The case $t = 2m$ is impossible to attain in an embedding in a surface since the triangles would close-up and the rotation at that vertex would consist of more than one cycle. If $t \leq 2m - 2$ then $\phi(x) \leq (2m + 1)/3$ where equality is attained only if $t = 2m - 2$ and the remaining four faces are quadrilaterals. This solution is indeed possible by our patchwork construction in the first half of the proof. In the remaining case ($t = 2m - 1$) we have $2m - 1$ triangular faces and 3 other faces. The triangular faces are necessarily consecutive in the rotation around $x$, since two of the neighbors of $x$ are not in triangles with $x$.

There are 4 sub-cases, concerning the number of quadrilateral faces $q = a_4(x)$. We may have $0 \leq q \leq 3$. By an arithmetical argument we rule out the cases $q = 0$ and $q = 1$. Case $q = 3$ is impossible, since $n > 2$ and one face has two edges projecting to $C_{2n}$. This leaves us with $q = 2$ and the remaining face either pentagonal ($a_5(x) = 1$) or hexagonal ($a_6(x) = 1$). Indeed, if the remaining face has size greater than 6, the value $(2m + 1)/3$ cannot be attained. The value $a_6(x) = 1$ gives us exactly $\phi(x) = (2m + 1)/3$. The only way that $a_5(x) = 1$ this could occur is to have a string of $2m - 1$ triangles ended on each side by a quadrilateral and
Figure 2: ... a string of \(2m - 1\) triangles ended on each side by a quadrilateral and the pentagonal face at \(x\).

Figure 3: Case \(m = 3\). The three triangles indicate the patchwork that was used for embedding \(K_{3,3,3} \times K_2\). The three thick edges mark the \(3\) selected quadrilaterals and the black triangle comes in two copies to complete the new patchwork of the embedded \(K_{3,3,3} \times K_2\).

The pentagonal face at \(x\) has both edges, say \(xy\) and \(xz\) projecting on \(C_{2m}\). But this is impossible, since the shortest path from \(y\) to \(z\) not using edge \(xy\) and/or \(xz\) has length \(4\). \(\Box\)

**Theorem 2.** The genus of \(K_{m,m,m} \times C_2, m \geq 1\) is given by the formula:

\[
\gamma(K_{m,m,m} \times C_2) = \gamma(K_{m,m,m} \times K_2) = 1 - 2m + m^2 = (m - 1)^2
\]

**Proof.** It is easy to see that the two graphs have the same genus embedding and hence consider \(K_2\) instead of \(C_2\). The proof is simpler but analogous to the proof of Theorem 1. In the construction we only need one patchwork. The surface is composed of two surfaces \(S_j\) joined by \(m\) tubes, hence, it has genus \((m - 1)^2\). The converse is easy since each vertex must necessarily contribute only \(2m - 1\) triangles, and \(2\) additional quadrilaterals is the best one can hope for. \(\Box\)

**Theorem 3.** In general the genus of \(K_{m,m,m} \times C_4\) is bounded as follows:

\[
[2m^2 - 5m/2 + 1] \leq \gamma(K_{m,m,m} \times C_4) \leq 1 + 2m(m - 1) = 2m^2 - 2m + 1.
\]

In particular,
1. for $m = 1$ the genus is given by
\[ \gamma(K_{1,1,1} \times C_4) = 1 \]

2. for $m = 2$ the genus is given by
\[ \gamma(K_{2,2,2} \times C_4) = 5 \]

3. for $m = 3$ the genus is given by
\[ \gamma(K_{3,3,3} \times C_4) = 12 \]

Proof. The upper bound $1 + 2m(m - 1)$ is obtained from construction of Theorem 1. The lower bound also follows from the argument in the proof of Theorem 1. Namely, here we cannot rule out the possibility that $\phi_0 = (m + 1)/2 + (2m - 1)/12 = (8m + 5)/12$ that would arise if $2m - 1$ triangles and $3$ quadrilaterals are incident with each vertex. For $m = 1$ the two bounds coincide. For $m = 2$ the genus is between $4$ and $5$ and one can easily check that no genus $4$ orientable embedding exists. For $m = 3$ the lower bound is $[11, 5] = 12$. In order to lower the upper bound to $12$ we may use the fact that $K_{m,m,m} \times C_4$ is isomorphic to $K_{m,m,m} \times K_2 \times K_2$. We start with the genus embedding of $K_{m,m,m} \times K_2$ described in the previous Theorem. It contains a patchwork consisting of $2$ triangles and $3$ quadrilaterals. Using this patchwork one can produce an embedding of $K_{m,m,m} \times K_2 \times K_2$ that has $56$ triangular and $30$ quadrilateral faces and is therefore genus $12$ embedding. The same idea could be explored for more general values of $m$. It would slightly improve the upper bound at least for $m$ that is divisible by $3$. □

References


