
MINIMAL DECOMPOSITION OF INDEFINITE HYPERGEOMETRIC SUMS

S. A. Abramov       M. Petkovšek

ISSN 1318-4865

April 11, 2001

Ljubljana, April 11, 2001
Abstract

We present an algorithm which, given a hypergeometric term \( T(n) \), constructs hypergeometric terms \( T_1(n) \) and \( T_2(n) \) such that \( T(n) = T_1(n + 1) - T_1(n) + T_2(n) \), and \( T_2(n) \) is minimal in some sense. This solves the decomposition problem for indefinite sums of hypergeometric terms: \( T_1(n + 1) - T_1(n) \) is the “summable part” and \( T_2(n) \) the “non-summable part” of \( T(n) \).

1 Introduction

A sequence \( T(n) \) is a hypergeometric term (or simply a term) if the ratio \( T(n + 1)/T(n) \) is a rational function of \( n \). We call this function the certificate of \( T \).

The well-known Gosper’s algorithm [4] solves the problem of indefinite hypergeometric summation: Given a hypergeometric term \( T(n) \), find another hypergeometric term \( T_1(n) \) such that

\[
T(n) = T_1(n + 1) - T_1(n),
\]

provided that such a term exists. If it does, and if \( T(k) \) and \( T_1(k) \) are defined for \( k = n_0, n_0 + 1, \ldots, n \), then we obtain from (1) the summation identity

\[
\sum_{k=n_0}^{n} T(k) = T(n) + T_1(n) - T_1(n_0).
\]

If no hypergeometric term \( T_1(n) \) satisfies (1), we can ask for two hypergeometric terms \( T_1(n) \) and \( T_2(n) \) such that

\[
T(n) = T_1(n + 1) - T_1(n) + T_2(n)
\]

and \( T_2(n) \) is minimal in some sense. We call this the decomposition problem for indefinite hypergeometric sums. For example, when \( T(n) \) is a rational function of \( n \), the algorithms given in [1, 2, 7] yield rational functions \( T_1(n) \) and \( T_2(n) \) such that (2) holds and the denominator of \( T_2 \) is of the least possible degree. These approaches are unified in [5].

Throughout the paper \( K \) is a field of characteristic zero, \( n \) an indeterminate, and \( E \) the shift operator on sequences with elements in \( K \) defined by \( ET(n) = T(n + 1) \). We write \( p \perp q \) to indicate that polynomials \( p, q \in K[x] \) are relatively prime. The leading coefficient of a rational function is the quotient of the leading coefficients of its numerator and denominator. A rational function is monic if its leading coefficient is 1.

Following [5] we introduce the notion of shift-equivalence among polynomials.

*Partially supported by Natural Sciences and Engineering Research Council of Canada under grant GRD215442-98, and by the French-Russian Lyapunov Institute, Project No 98-03.

†Partially supported by MZT RS under grant J2-8549.
**Definition 1** Polynomials \( p, q \in K[n] \) are shift-equivalent if \( p = E^k q \) for some \( k \in \mathbb{Z} \). In this case we write \( p \sim q \). A monic rational function is shift-homogeneous if all nonconstant irreducible monic factors of its numerator and denominator belong to the same shift-equivalence class.

By grouping together shift-equivalent irreducible monic factors of its numerator and denominator every rational function can be written in the form

\[
R(n) = z R_1(n) R_2(n) \cdots R_k(n)
\]  

where \( z \in K, k \geq 0 \), each \( R_i \) is a shift-homogeneous rational function, and any two nonconstant monic irreducible factors \( p \) of \( R_i \) and \( q \) of \( R_j \) are pairwise shift-in equivalent whenever \( i \neq j \). We call (3) a shift-homogeneous factorization of \( R \). It is clear that the shift-homogeneous factorization is unique up to the order of the factors.

**Definition 2** Let \( r, s \in K[n] \). If \( r \perp E^k s \) for all \( k \in \mathbb{Z} \), then the rational function \( r/s \) is shift-reduced. If \( r \perp E^k r \) for all \( k \in \mathbb{Z} \setminus \{0\} \) then the polynomial \( r \) is shift-free.

Now we define a normal form for rational functions which reveals the shift structure of its factors. We need this normal form to define a measure of complexity of a hypergeometric term.

**Definition 3** Let \( R \in K(n) \setminus \{0\} \) be a nonzero rational function. If \( z \in K \) and monic polynomials \( r, s, u, v \in K[n] \) satisfy \( R = z \cdot \frac{r}{s} \cdot \frac{u}{v} \) where \( r/s \) is shift-reduced, \( F = u/v \) and \( u \perp v \), then \((z, r, s, u, v)\) is a rational normal form (RNF) of \( R \). The rational function \( r/s \) is the kernel of this RNF.

In [8] it is shown that every nonzero rational function has an RNF, and that regarding uniqueness we have the following results:

**Lemma 1** Let \( R \in K(n) \) be shift-homogeneous. If \((1, r, s, u, v)\) and \((1, r_1, s_1, u_1, v_1)\) are two RNF’s of \( R \) then \( r = r_1 = 1 \) and \( \deg s = \deg s_1 \), or \( s = s_1 = 1 \) and \( \deg r = \deg r_1 \).

**Theorem 1** Let \((z, r, s, u, v)\) and \((z', r', s', u', v')\) be two RNF’s of \( R \in K(n) \setminus \{0\} \). Then

(i) \( z = z' \),
(ii) \( \deg r = \deg r' \) and \( \deg s = \deg s' \),
(iii) there is a one-to-one correspondence \( f \) between the multisets of nonconstant irreducible monic factors of \( r \) and \( r' \) such that \( p \sim f(p) \) for all \( p \mid r \),
(iv) there is a one-to-one correspondence \( g \) between the multisets of nonconstant irreducible monic factors of \( s \) and \( s' \) such that \( q \sim g(q) \) for all \( q \mid s \).

Now we can formulate the decomposition problem for indefinite hypergeometric sums in the following way:

Given a hypergeometric term \( T \), find hypergeometric terms \( T_1, T_2 \) such that

\[
T = (E - 1) T_1 + T_2
\]

and the certificate \( ET_2/T_2 \) has an RNF

\[
(z, r, s, u, v)
\]

with \( v \) of minimal degree.

This formulation agrees with the decomposition problem for indefinite sums of rational functions [1, 2, 7] because if \( T_2 \in K(n) \) then \( r = s = 1 \) and \( v \) is the denominator of \( T_2 \).

**Definition 4** If \((z, r_1, s_1, u_1, v_1)\) and \((z, r_2, s_2, u_2, v_2)\) are two RNF’s for the same rational function we will say that the former is simpler than the latter if \( \deg v_1 < \deg v_2 \).
If the terms \( T, T_1, T_2 \) satisfy (4) then they are similar, i.e., there exist \( R_1, R_2 \in K(n) \) such that \( T_1 = R_1 T \) and \( T_2 = R_2 T \) (cf. [6, Prop. 5.6.2]).

**Definition 5** A rational function \( F \in K(n) \) is adequate for a hypergeometric term \( T \) if the certificate \( ET/T \) has an RNF with \( F \) as its kernel.

Let \( T, T_1, T_2 \) satisfy (4). By the similarity of these terms each rational function that is adequate for one of them is adequate for the other two.

We will describe a solution of the decomposition problem in three stages:

(A) We describe an algorithm which, given a term \( T \), constructs terms \( T_1, T_2 \) similar to \( T \), and a rational function \( F \) adequate for \( T, T_1, T_2 \), such that if one restricts attention to RNF’s with kernel \( F \) then \( T_1, T_2 \) solve the decomposition problem.

(B) We prove that for any other rational function \( F' \) adequate for \( T \) and for any term \( T'_1 \) similar to \( T \), the RNF (with kernel \( F' \)) of \( ET'_2/T'_2 \) where \( T'_2 = T - (E - 1)T'_1 \) is not simpler than the RNF of the certificate of \( T_2 \) constructed by the algorithm in (A).

(C) Finally we show how one can “improve” \( T_2 \) without changing \( \text{deg } v \) in such a way that \( \text{deg } u \) in (5) is not too large.

## 2 Part (A): An algorithm

**Definition 6** Let \( n_0 \in \mathbb{Z} \) and \( D, U \in K(x) \). The triple \((D, U, n_0)\) regularly describes a term \( T(n) \) if for all integer \( n \geq n_0 \)

- \( D(x), U(x) \) have neither a pole nor a zero at \( x = n \),
- \( T(n) = U(n) \prod_{k=n_0}^{n-1} D(k) \).

Let \((D, U, n_0)\) regularly describe a term \( T \), \( n_1 > n_0 \), and

\[
V(x) = U(x) \frac{T(n_1)}{U(n_1)} = U(x) \prod_{k=n_0}^{n_1-1} D(k).
\]

Then it is evident that \((D, V, n_1)\) regularly describes \( T \).

**Lemma 2** Let the triples \((D, U, n_0)\) and \((D, U_1, n_0)\) regularly describe (similar) terms \( T \) and \( T_1 \). Then the certificate of the term \( T_2 = T - (E - 1)T_1 \) is

\[
\frac{DU_2}{U_2} = D\frac{EU_2}{U_2} + U_1.
\]

where

\[
U_2 = U - D(EU_1) + U_1.
\]

**Proof:** For all integer \( n \geq n_0 \) we have

\[
T_2(n) = U(n) \prod_{k=n_0}^{n-1} D(k) - (E - 1)U_1(n) \prod_{k=n_0}^{n-1} D(k)
\]

\[
= U(n) \prod_{k=n_0}^{n-1} D(k) - (EU_1(n)) \prod_{k=n_0}^{n-1} D(k) + U_1(n) \prod_{k=n_0}^{n-1} D(k)
\]

\[
= (U(n) - (EU_1(n))D(n) + U_1(n)) \prod_{k=n_0}^{n-1} D(k).
\]

It follows that the values of \( ET_2/T_2 \) are equal to the values of (6) for all integer \( n \geq n_0 \). This proves the claim.
Lemma 3 Let $D, U \in K(n)$, $D = d_1/d_2$, $U = u_1/u_2$ where $d_1, d_2, u_1, u_2 \in K[n]$, and $D$ is shift-reduced. Then there is $U_1 \in K(n)$ such that substituting it into the right-hand side of (7) one gets

$$U_2 = \frac{v_1}{(E^{-1}d_1)d_2v_2},$$

where $i, j \in \{0, 1\}$, $v_1, v_2 \in K[n]$, and for any irreducible $p \in K[n]$ dividing $v_2$ and any $h \in \mathbb{Z}$ the following conditions are satisfied:

$$E^h q | v_2 \Rightarrow h = 0,$$  

(9)

$$E^h q | d_1 \Rightarrow h < 0,$$  

(10)

$$E^h q | d_2 \Rightarrow h > 0.$$  

(11)

(in words, (9) means that $v_2$ is shift-free).

Proof: Let $q$ be an irreducible in $K[n]$ and $u_2 = u_2q^h$ where $q \perp u_2$ and $k > 0$. Then there are $a, b \in K[n]$ such that

$$U = \frac{a}{u_2} + \frac{b}{q^k}.$$  

(12)

We distinguish two cases.

a) There is an integer $h \leq 0$ such that $E^h q | d_2$. Let $U_1' = E^{-1}(b/(Dq^k))$. Then $D(EU_1') = b/q^k$, so $U - D(EU_1') + U_1'$ can be written as

$$\frac{c_0}{u_2} + \frac{c_1}{E^{-1}d_1} + \frac{c_2}{(E^{-1}q)^l}$$

where $l \leq k$, $c_0, c_1, c_2 \in K[n]$.

b) There is an integer $h \geq 0$ such that $E^h q | d_1$. Let $U_1' = -b/q^k$. Then $U - D(EU_1') + U_1'$ can be written as

$$\frac{c_0}{u_2} + \frac{c_1}{d_2} + \frac{c_2}{(Eq)^l}$$

where $l \leq k$, $c_0, c_1, c_2 \in K[n]$.

Since $D$ is shift-reduced, at most one of the cases a), b) can occur. Repeating these steps if necessary (using $U_1'^m, U_1'^n, \ldots$) we obtain a rational function $U = D(EU_1' + U_1'^m + \cdots) + (U_1'^n + U_1'^n + \cdots)$ whose denominator is divisible by at most one polynomial of the form $E^\gamma q$. If such a $\gamma$ exists then for $p = E^\gamma q$ we have (9), (10), and (11). Similarly we can proceed with the irreducible factors of $u_2$ that are different from $E^\gamma q, \sigma \in \mathbb{Z}$.

Suppose that $u_2$ is not shift-free. If there is an integer $h > 0$ such that $E^h q | u_2$ for some irreducible $q$ and if (12) holds, then we can transform $U$ as it is described in a). Similarly we can use the way described in b) if we have $h < 0$.

Lemma 4 Let $T$ be a term with the certificate $D_{\alpha U_1}E_{U_2}$, where $D = d_1/d_2$ is a shift-reduced rational function and $U = u_1/u_2 \in K(n)$. Let $U_1, U_2$ be rational functions that exist by Lemma 3. Then there exists a term $T_1$, similar to $T$, such that the term $T_2 = T - (E - 1)T_1$ has $D_{\alpha U_1}E_{U_2}$ as its certificate.

Proof: Let $n_0$ be an integer such that $D(n), U(n), U_1(n)$ have no pole or zero for integer $n \geq n_0$ and the value $T(n)$ is defined for $n \geq n_0$. Set $\alpha = T(n_0)/U(n_0)$. By (7) we have

$$\alpha U_2 = \alpha U - D(E(\alpha U_1)) + \alpha U_1.$$  

The triple $(D, \alpha U, n_0)$ regularly describes $T$, the triple $(D, \alpha U_1, n_0)$ in turn regularly describes the term $T_1(n) = \alpha U_1(n) \prod_{k=n_0}^{n-1} D(k)$. Then by Lemma 2 the term $T_2 = T - (E - 1)T_1$ has the certificate $D_{\alpha U_1}E_{U_2} = D_{\alpha U_2}E_{U_2}$.

\[\Box\]
Using Lemmas 3 and 4 one obtains the terms \( T_1 \) and \( T_2 = T - (E - 1)T_1 \) with certificates in the form \( D^E_{U_1} \) and \( D^E_{U_2} \), respectively. The certificate of \( T_2 \) can be rewritten in a simpler form: suppose that \( U_2 \) has the form (8), then we can remove the factors \( (E^{-1}d_j)^i, d_j^i, i, j \in \{0, 1\} \) from the denominator of \( U_2 \), since

\[
\frac{D^E_{U_2}}{U_2} = \frac{d_1 \left( \frac{E^{-1}d_1}{d_2} \right)^i}{d_2 \left( \frac{E^{-1}d_1}{d_2} \right)^j} \frac{E^{v_1}}{v_2}
\]

Setting

\[
F = \frac{d_1 \left( \frac{E^{-1}d_1}{d_2} \right)^i}{d_2 \left( \frac{E^{-1}d_1}{d_2} \right)^j}, \quad V = \frac{v_1}{v_2},
\]

we get the certificate of \( T_2 \) in the form \( F^E V \). Reformulating properties (9), (10), (11) in terms of \( F \) and \( V \), we have the following theorem.

**Theorem 2** Let \( T \) be a term. Then it is possible to find a term \( T_1 \) similar to \( T \), and a shift-reduced rational function \( F = f_1/f_2 \) that is adequate for \( T, T_1 \), such that the certificate of the term \( T_2 = T - (E - 1)T_1 \), written in the RNF with \( F \) as the kernel:

\[
F^E V, \quad V = \frac{v_1}{v_2}, \quad v_1 \perp v_2,
\]

has the following properties:

**(Pa)** \( v_2 \) is shift-free;

**(Pb)** if \( p \) is an irreducible from \( K[n] \) such that \( p \mid v_2 \), then

\[
E^h p \mid f_1 \Rightarrow h < 0, \quad \text{(13)}
\]

\[
E^h p \mid f_2 \Rightarrow h > 0. \quad \text{(14)}
\]

As the proofs of Lemmas 3 and 4 are constructive, we can now describe an algorithm to compute the terms \( T_1, T_2 \) mentioned in Theorem 2. In the case a) of the proof of Lemma 3 we considered irreducible \( q \) and integer \( h \) such that \( q \mid v_2, E^h q \mid d_2, h \leq 0 \). All the \( q \)'s (say \( q_1, \ldots, q_\epsilon \)) that relate to the minimal possible \( h \) can be considered together. Using a resultant and gcd techniques, we can find the minimal value of \( h \) along with \( q^1 \equiv q_1 \equiv \ldots \equiv q^\epsilon \mid v_2, v_1, \ldots, v_\epsilon > 0 \). After this we can compute \( \bar{q} = q_1^{\mu_1} \ldots q_\epsilon^{\mu_\epsilon} \), where \( \mu_1, \ldots, \mu_\epsilon \) are maximal possible values such that \( q_1^{\mu_1} \ldots q_\epsilon^{\mu_\epsilon} \mid v_2 \). We can use the following simple algorithm \textit{pump}:

**input:** \( f, g \in K[n], f \mid g \);

**output:** \( f', g \in K[n], f \mid g, \bar{f} \perp \bar{g} \);

\[
f' := f; \quad \bar{g} := g/f; \quad \text{repeat}
\]
 \[
d := \gcd(f', \bar{g});
\]
 **\( f' := f'd; \bar{g} := \bar{g}/d; \text{ until } \deg d = 0 \).**

We can compute \((q, q_2) = \text{pump}(q, u_2)\) and then use the partial-fraction decomposition

\[
U = \frac{\bar{a}}{\overline{w_2}} + \frac{\bar{b}}{\overline{\bar{q}}}
\]

where \( \bar{a}, \bar{b} \in K[n] \), in place of decomposition (12). We can similarly proceed in the case b) of the proof of Lemma 3. Thus we finally arrive at the following algorithm \textit{haldecomp}:

**input:** \( D = \frac{d_1}{d_2}, U = \frac{u_1}{u_2} \), where \( d_1 \perp d_2 \) and \( D \) is shift-reduced.

**output:** \( U_1, F, V \in K(n) \) such that the term with certificate \( D^E_{U_1} \) has decomposition \( T = T_2 + (E - 1)T_1 \).
where \( T_1, T_2 \) have certificates \( D_{U_1}^{E_1} \) and \( F_{U_2}^{E_2} \), resp., with \( F, V \) satisfying Pa, Pb of Theorem 2.

\[
U_1 := 0; \quad U_2 := U;
N_1 := -1; \quad N_2 := 1; \quad M := -1;
R_1(m) := \text{Res}_u(u_2(x + m), d_1(x));
R_2(m) := \text{Res}_u(u_2(x + m), d_2(x));
R(m) := \text{Res}_u(u_2(x + m), u_2(x));
\]

if \( R_1(m) \) has some nonnegative integer root
   then \( N_1 := \max\{m : m \in \mathbb{Z}, R_1(m) = 0\} \)
fi;
if \( R_2(m) \) has some nonpositive integer root
   then \( N_2 := \min\{m : m \in \mathbb{Z}, R_2(m) = 0\} \)
fi;
if \( R(m) \) has some positive integer root
   then \( M := \max\{m : m \in \mathbb{Z}, R(m) = 0\} \)
fi;
\[ N_1 = \max\{N_1, M\}; \]

for \( h = N_1 \) downto 0 do
\[
q := \gcd(u_2, E^{-h}d_1);

\]

\[
t := u_2/q;
q := q \gcd(t, E^{-h}t);
(q, u_2) := \gcd(q, u_2);
\]

represent \( U_2 \) in the form \( U_2 = \frac{a}{u_2} + \frac{b}{t}, \quad a, b \in K[n] \);
\[
U_1' := -\frac{b}{q};
U_2 := U_2 - D(EU_1') + U_1'; \quad U_1 := U_1 + U_1'
\]
don;

for \( h = N_2 \) to 0 do
\[
q := \gcd(u_2, E^{-h}d_2);

t := u_2/q;
q := q \gcd(t, E^{-h}t);
(q, u_2) := \gcd(q, u_2);
\]

represent \( U_2 \) in the form \( U_2 = \frac{a}{u_2} + \frac{b}{t}, \quad a, b \in K[n] \);
\[
U_1' := E^{-1}(b/(Dq));
U_2 := U_2 - D(EU_1') + U_1'; \quad U_1 := U_1 + U_1'
\]
don;
\[
v_1 := \text{numerator}(U_2); \quad v_2 := \text{denominator}(U_2);
\]

if \( E^{-1}d_1 | v_2 \)
   then \( v_2 := v_2/(E^{-1}d_1); \quad f_1 := E^{-1}d_1 \)
else \( f_1 := E^{-1}d_1 \)
fi;
if \( d_2 | v_2 \)
   then \( v_2 := v_2/d_2; \quad f_2 := Ed_2 \)
else \( f_2 := d_2 \)
fi;
\[
F := \frac{f_1}{f_2}; \quad V := \frac{u_1}{v_2};
\]

If \( n_0 \in \mathbb{Z} \) is such that \( D(x), U(x), U_1(x), F(x), V(x) \) have neither a pole nor a zero for \( x \geq n_0 \), then the proof of Lemma 4 allows one to obtain regular descriptions of terms \( T \) and \( T_1 \). It is also possible to get a regular description of \( T_2 \). Indeed, it is sufficient to find a constant \( \gamma \) such that \( T_2(n) = \gamma V(n) \prod_{n_0} F(k) \).

For \( n = n_0 \) we have
\[
\gamma V(n_0) = T_2(n_0) = T(n_0) - T_1(n_0 + 1) + T_1(n_0) = \alpha \left( U(n_0) - U_1(n_0 + 1)D(n_0) + U_1(n_0) \right)
\]
where \( \alpha = T(n_0)/U(n_0) \). This gives \( \gamma \).
So, after applying $hyde\text{comp}$ we can find regular descriptions of the terms $T_1$ and $T_2$. We will prove in the rest of the paper that these terms give a solution of the decomposition problem.

**Theorem 3** Let $T$, $T_1$, $T'_1$ be similar terms, $T_2 = T - (E - 1)T_1$, $T'_2 = T - (E - 1)T'_1$, and $F = f_1 / f_2$ a shift-reduced rational function adequate for these terms. Let $ET_2 / T_2 = F^{EW}$ where $F, V \in K[n]$ have properties $\text{Pa}$ and $\text{Pb}$ of Theorem 2, and $ET'_2 / T'_2 = F^{WV}$. If $V = v_1 / v_2$ and $V' = v'_1 / v'_2$ where $v_1, v_2, v'_1, v'_2 \in K[n]$ and $v_1 \perp v_2$ then $\deg v_2 \leq \deg v'_2$.

**Proof:** We have

$$T'_2 = T_2 - (E - 1)(T'_1 - T_1).$$

Suppose that the certificate of $T'_1 - T_1$ is equal to $F(EW) / W$ where $W = w_1 / w_2$ and $w_1 \perp w_2$. Then, by (7),

$$\frac{v'_1}{v'_2} = \frac{v_1}{v_2} - \frac{f_1 E w_1}{f_2 E w_2} + \frac{w_1}{w_2}.$$  

(15)

Consider an arbitrary irreducible $p \in K[n]$ such that $p | v_2$. We set

$$k = \max \{ \alpha : p^\alpha | v_2 \}$$

and claim that $E^l p^k | v'_2$ for some $l \in \mathbb{Z}$. Since the pair $F, V$ has property $\text{Pa}$, this claim will imply the statement of the theorem. Suppose that $p^k$ does not divide $v'_2$. Equation (15) implies that $v_2$ and hence $p^k$ divides the lcm of $v'_2$, $f_2 E w_2$, and $w_2$. By (14) we have $p \perp f_2$, therefore $p^k | Ew_2$ or $p^k | w_2$.

Let $p^k | Ew_2$. Then

$$E^{-1} p^k | w_2.$$  

(16)

Set $l = \min \{ m : E^m p^k | w_2 \}$. Apparently $E^l p^k$ does not divide $Ew_2$. It follows from (16) that $l \leq -1$; together with (14) this gives $E^l p \perp f_2$. As $v_2$ is shift-free and $p | v_2$, it follows that $E^l p^k$ does not divide $v_2$. Therefore (15) implies

$$E^l p^k | v'_2.$$  

(17)

Let $p^k | w_2$. Then

$$E^l p^k | Ew_2.$$  

(18)

Set $l = \max \{ m : E^m p^k | Ew_2 \}$. Apparently $E^l p^k$ does not divide $w_2$. It follows from (18) that $l \geq 1$; together with (13) this gives $E^l p \perp f_1$. Therefore (15) implies (17) in this case as well. \qed

**Corollary 1** Let $F, U, S_1, S_2 \in K(n)$ where $F$ is shift-reduced. Let rational functions

$$V_1 = U - FES_1 + S_1, \quad V_2 = U - FES_2 + S_2$$

be such that the pairs $F, V_1$ and $F, V_2$ have properties $\text{Pa}$ and $\text{Pb}$ of Theorem 2. Then the degrees of the denominators of $V_1$ and $V_2$ are equal.

3 Part (B): A verification

If both rational functions $F_1, F_2$ are adequate for a term $T$ then there exists $G \in K(n)$ such that

$$\frac{F_1}{F_2} = \frac{EG}{G}.$$  

(19)

Indeed, for some $U_1, U_2 \in K(n)$ we have

$$F_1 EU_1 = F_2 EU_2,$$

and therefore $G = U_2^{-1} U_1$. The case where $G \in K[n]$ is especially interesting for us.
Theorem 4 Let $F_1, F_2$ be rational functions that are adequate for a term $T$. Let (19) hold with $G \in K[n]$. Let $V$ be such that the pair $F_1, V$ has properties £a and £b. Then the denominator of $V$ is coprime with $G$ and the pair $F_2, G \cdot V$ has properties £a and £b.

Proof: First we prove that the denominator of $V$ is coprime with $G$. If they have a common irreducible factor $p$ then the set $\{ \nu : E^\nu p \, | \, G \}$ is a non-empty finite set. Suppose that $m, M$ are, resp., the minimal and the maximal elements of this set. Write

$$W = \frac{G}{EG} = \frac{F_2}{F_1} = \frac{w_1}{w_2} \quad w_1 \perp w_2.$$ 

Then $E^{M+1} p \, | \, w_2$ and $E^m p \, | \, w_1$. We have $F_2 = W F_1$. As $p$ divides the denominator of $V$ and the pair $F_1, V$ has properties £a and £b, the numerator of $F_1$ is not divisible by $E^{M+1} p$ since $M + 1 > 0$. Similarly the denominator of $F_1$ is not divisible by $E^m p$ since $m \leq 0$. Therefore the numerator of $F_2$ is divisible by $E^m p$ while the denominator of $F_2$ is divisible by $E^{M+1} p$. But $F_2$ is shift-reduced by Definition 3(ii), a contradiction.

Now we prove that the pair $F_2, G \cdot V$ has properties £a and £b. We have

$$F_2 = \frac{G}{EG} F_1$$

and the pair $F_2, G \cdot V$ has property £a because the denominator of $G \cdot V$ divides the denominator of $V$. Now we shall be concerned with £b. Let $p$ be an irreducible from $K[n]$ that divides the denominator of $G \cdot V$ and thereby divides the denominator of $V$. Let $E^h p$, $h \leq 0$, divide the denominator of $F_2$. Then $E^h p$ does not divide the denominator of $F_1$ since the pair $F_1, V$ has properties £a and £b. The equality $(EG) F_2 = GF_1$ implies that $E^h p \, | \, EG$. Set $h_0 = \min \{ \nu : E^\nu p \, | \, EG \}$. Then $h_0 \leq h \leq 0$ and $E^{h_0-1} p \, | \, G$, but $E^{h_0-1} p$ does not divide $EG$. The denominator of $F_1$ is not divisible by $E^{h_0-1} p$ since the pair $F_1, V$ has properties £a and £b. Therefore $E^{h_0-1} p$ divides the numerator of $F_2$. But as $E^h p$ divides the denominator of $F_2$, this contradicts the fact that $F_2$ is shift-reduced.

Similarly it can be shown that $E^h p$, $h \geq 0$, cannot divide the numerator of $F_2$. □

Lemma 5 Let $F, F_1, U, U_1 \in K(n)$, $G \in K[n]$ be such that $F / F_1 = EG / G$, $G \in K[n]$ and $F \, | \, G U_1 = F_1 \, | \, G U_1$. Then there exists $\overline{G} \in K[n]$ such that $\overline{G} U = U_1$ and for any $S \in K(n)$ we have

$$\overline{G}(U - FES + S) = U_1 - F_1 E(\overline{G}S) + \overline{G}S.$$  

Proof: We have

$$\frac{E(U^{-1}U_1)}{U^{-1}U_1} = \frac{EG}{G}.$$ 

It follows from this that there exists $\alpha \in K$ such that $U^{-1} U_1 = \alpha G$. Set $\overline{G} = \alpha G$. We get

$$\frac{EG}{G} F_1 = F, \quad U_1 = \overline{G} U.$$ 

Substituting $U_1$ for $\overline{G} U$ and $(EG / G) F_1$ for $F$ in $\overline{G} U = (\overline{G} F) ES + \overline{G} S$ gives $U_1 = F_1 E(\overline{G}S) + \overline{G}S$. □

Theorem 5 Let $F_1, F_2$ be rational functions that are adequate for a term $T$. Let $U_1, U_2, R \in K(n)$ be such that

$$\frac{F_1}{U_1} = \frac{F_2}{U_2} = R. \quad (20)$$

For $S_1, S_2 \in K(n)$, let

$$V_1 = U_1 - F_1 E S_1 + S_1, \quad V_2 = U_2 - F_2 E S_2 + S_2 \quad (21)$$

be such that the pairs $F_1, V_1$ and $F_2, V_2$ have properties £a and £b. Then the denominators of $V_1$ and $V_2$ have equal degrees.
Proof: First of all we show that there exists a rational function \( \alpha/\beta \), \( \alpha \perp \beta \), such that for the rational functions

\[
F_0 = \frac{a}{b}, \quad F_{-1} = \frac{E^{-1}a}{b}, \quad F_{-2} = \frac{a}{E^b}, \quad F_{-3} = \frac{E^{-1}a}{E^b}
\]

the equalities

\[
F_i = \frac{EG_i}{G_i}, \quad F_i = \frac{EG_i'}{G_i'}, \quad G_i, G_i' \in K[n].
\]

hold for \( i = -1, -2, -3 \). It is sufficient to prove the theorem for shift-homogeneous \( F_1, F_2 \) which belong to the same shift-homogeneous class. Then, by Lemma 1, either both \( F_1 \) and \( F_2 \) are polynomials, or both \( F_1 \) and \( F_2 \) are reciprocals of polynomials. By Theorem 1(ii) we have

\[
F_1 = \prod_{i=1}^{\tau} E^{b_i}p, \quad F_2 = \prod_{i=1}^{\tau} E^{d_i}p,
\]

in the former case, and

\[
F_1 = \frac{1}{\prod_{i=1}^{\tau} E^{b_i}p}, \quad F_2 = \frac{1}{\prod_{i=1}^{\tau} E^{d_i}p},
\]

in the latter, where \( p \in K[n] \) is irreducible. In case of (24), set

\[
a = \prod_{i=1}^{\tau} E^{\max(b_i, d_i) - 1}p, \quad b = 1,
\]

and in case of (25), set

\[
a = 1, \quad b = \prod_{i=1}^{\tau} E^{\min(b_i, d_i) - 1}p.
\]

It is easy to see that if \( F_0, F_{-1}, F_{-2}, F_{-3} \) are defined as in (22) then the equalities (23) hold for some polynomials \( G_i, G_i' \).

Considering the RNF of \( R \) with the kernel \( \alpha/\beta \) and using algorithm \( \text{hydecomp} \) we can get \( i, -3 \leq i \leq 0 \), and \( F, U, V, S \in K[n] \) such that

- \( F = F_i \),
- \( R = F E^{U_{i-1}} U = \frac{u_1}{u_2}, \quad u_1 \perp u_2; \)
- \( V = U - F E S + S, \)
- the pair \( F, V \) has properties \( \text{Pa} \) and \( \text{Pb} \).

Set

\[
G_i = G_i', \quad G_i'' = G_i''
\]

for the computed \( i \). By Lemma 5 we have a polynomial \( G_i' \) such that

\[
G_i' V = G_i' (U - F E S + S) = U_1 - F_1 E (G_i' S) + G_i' S.
\]

By Theorem 4 the pair \( F_1, U_1 - F_1 E (G_i' S) + G_i' S \) has properties \( \text{Pa} \) and \( \text{Pb} \) and the degree of denominator of \( G_i' V \) is equal to the degree of the denominator of \( V \). By Corollary 1 we have that the denominator of \( V \) is of the same degree as the denominator of \( V_1 \), and similarly for the degrees of the denominators of \( V_0 \) and \( V_2 \). The claim follows.

\[ \Box \]

**Theorem 6** Let \( T, T_1, T_1' \) be similar terms. Let the certificates of the terms \( T_2 = T - (E - 1)T_1, T_2' = T - (E - 1)T_1' \) be written in the form

\[
F E V, \quad F' E V', \quad F E V, \quad F' E V'
\]

with shift-reduced \( F, F' \). Let the pair \( F, V \) have properties \( \text{Pa} \) and \( \text{Pb} \). Then the degree of the denominator of \( V \) is less than or equal to the degree of the denominator of \( V' \).

Proof: It is possible to find \( U, S, S' \in K(n), n_0 \in \mathbb{Z} \) and \( \alpha, \beta \in K \) such that the triples \( (F, S, n_0), (F, \alpha V, n_0) \) regularly describe the terms \( T, T_1, T_2 \) and the triples \( (F, S', n_0), (F, \beta V', n_0) \) regularly describe the terms \( T_1', T_2' \). The claim follows from Lemma 2 and Theorems 3, 5.

\[ \Box \]
4 Part (C): Decreasing the degree of the numerator

So, the denominator of $V$ has the minimal possible degree. How to reduce the degree of the numerator of $V$ to a tolerable size? Recall that when one solves the decomposition problem for indefinite sums of rational functions, it is always possible to have the degree of the numerator less than the degree of the denominator (this is because any rational function is the sum of a polynomial and a proper rational function, and the equation $(E - 1)y = u$ has a polynomial solution for any $u \in K[x]$).

In the hypergeometric case the situation is not so simple. Consider a term which is regularly described by a triple $(D, V, n_0)$, $V \in K[n]$, $D = d_1/d_2$, $d_1 \perp d_2$. Let’s try first to find a polynomial $S$ such that the polynomial

$$d_2V - d_1ES + d_2S$$

has a “reasonable” degree. Rewrite (26) as

$$P = d_2V - d_1(E - 1)S + (d_2 - d_1)S$$

and set

$$M = d_2V, \quad T = -d_1(E - 1)S + (d_2 - d_1)S.$$ 

Evidently $\deg (E - 1)S = \deg S - 1$, $lc((E - 1)S) = \deg S \cdot lcS$. By a judicious choice of $S$ we can cancel out some leading terms of $M$ if the degree of $M$ is large enough. The number of those terms depends first of all on the relation between $\deg(d_2 - d_1)$ and $\deg d_1$:

1. $\deg(d_2 - d_1) > \deg d_1$. Then $\deg d_2 > \deg d_1$ and $\deg T = \deg d_2 + \deg S$, $lcT = lc d_2 \cdot lc S$. We can transform $M$ to $P$ of degree $< \deg d_2$.

2. $\deg(d_2 - d_1) = \deg d_1$. Then $\deg d_2 \leq \deg d_1$ and $\deg T = \deg d_1 + \deg S$, $lcT = lc (d_2 - d_1) \cdot lc S$. We can transform $M$ to $P$ of degree $< \deg d_1$.

3. $\deg(d_2 - d_1) < \deg d_1$. Then $\deg d_2 = \deg d_1$, $lc d_2 = lc d_1$.

3a. $\deg(d_2 - d_1) < \deg d_1 - 1$. Then the coefficients of $x^{\deg d_1 - 1}$ in $d_1$ and $d_2$ are equal. We have $\deg T = \deg d_1 + \deg S - 1$, $lcT = -lc d_1 \cdot lcS \cdot \deg S$. We can transform $M$ to $P$ of degree $\sigma < \deg d_1$. (More, if $\sigma = \deg(d_2 - d_1)$, then we can by an additional elimination transform $P$ to $P'$ of degree $< \deg (d_2 - d_1)$.)

3b. $\deg(d_2 - d_1) = \deg d_1 - 1$. Then the coefficients of $x^{\deg d_1 - 1}$ in $d_1$ and $d_2$ are not equal. If $-lc d_1 \cdot \deg S + lc (d_2 - d_1) \neq 0$

then

$$\deg T = \deg d_1 + \deg S - 1 = \deg(d_2 - d_1) + \deg S,$$

$$lcT = lcS \cdot (-lc d_1 \cdot \deg S + lc (d_2 - d_1)).$$

In such a case if the equation

$$-lc d_1 \cdot X + lc(d_2 - d_1) = 0$$

has no integer root on the segment $[0; \deg M - \deg(d_2 - d_1)]$ or, equivalently, on the segment $[0; \deg M - \deg d_1 + 1]$, then we can transform $M$ to $P$ of degree $< \deg d_1 - 1$. If

$$-lc d_1 \cdot \tau + lc (d_2 - d_1) = 0$$

for an integer $\tau$ from the segment $[0; \deg M - \deg(d_2 - d_1)]$, then we can transform $M$ to $P$ of degree $< \deg(d_2 - d_1) + \tau + 1$, or, equivalently, $< \deg d_1 + \tau$.

Therefore we have the following
**Theorem 7** Let $M, d_1, d_2 \in K[n]$, $d_2 \neq 0$. Then it is possible to find $S \in K[n]$ such that the degree of $P = M - d_1 ES + d_2 S$ is less than

$$
\lambda = \begin{cases} 
deg d_2 & \text{if } \deg(d_2 - d_1) > \deg d_2, \\
\deg d_1 & \text{if } \deg(d_2 - d_1) = \deg d_2 \text{ or } \deg(d_2 - d_1) < \deg d_1 - 1, \\
\deg d_1 + \tau & \text{if } \deg(d_2 - d_1) = \deg d_1 - 1,
\end{cases}
$$

where in the last case $\tau$ is equal to $\text{lcm}(d_2 - d_1)/\text{lcm}d_1$ if this is a nonnegative integer and $-1$ otherwise.

Observe that if $d_1 = d_2 = 1$, then $\lambda = 0$ and $P = 0$. This is in agreement with the fact that the equation $(E - 1)S = V$ has a polynomial solution when $V \in K[n]$.

Thus we can find $S$ such that (26) is equal to a polynomial $P$ whose degree is bounded from above as described in Theorem 7. This implies that

$$
V(n) \prod_{k=\nu_0}^{\nu_1 - 1} D(k) - (E - 1)S(n) \prod_{k=\nu_0}^{\nu_1 - 1} D(k) \quad = \quad \frac{P(n)}{d_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} D(k) = \frac{P(n)}{d_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k + 1)}.
$$

Now suppose that $V$ is a rational function of the form $v_1/v_2$. This leads to the expression

$$
d_2v_1 - v_2d_1 ES + v_2d_2 S.
$$

We can use the described techniques, considering $v_2d_1, v_2d_2$ instead of $d_1, d_2$. We get $S, P$ such that

$$
d_2v_1 - v_2d_1 ES + v_2d_2 S = P,
$$
in other words

$$
\frac{v_1(n)}{v_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k)} - (E - 1)\left(S(n) \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k)}\right)
\quad = \quad \frac{P(n)}{v_2(n)d_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k)} = \frac{P(n)}{v_2(n)d_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k + 1)}.
$$

Hence we have proven

**Theorem 8** Let a term $T$ be regularly described by a triple $(d_1/d_2, v_1/v_2, n_0)$, i.e.,

$$
T(n) = \frac{v_1(n)}{v_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k)}.
$$

Then there exists a term $T_1$ of the form

$$
T_1(n) = S(n) \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{d_2(k)}.
$$

$S \in K[n]$, such that the term $T_2 = T - (E - 1)T_1$ is of the form

$$
\frac{P(n)}{v_2(n)} \prod_{k=\nu_0}^{\nu_1 - 1} \frac{d_1(k)}{f(k)},
$$

where $f(k)$ is either $d_2(k)$ or $d_2(k + 1)$ and $P$ is a polynomial whose degree is less than

$$
\lambda = \begin{cases} 
\deg v_2 + \deg d_2 & \text{if } \deg(d_2 - d_1) > \deg d_2, \\
\deg v_2 + \deg d_1 & \text{if } \deg(d_2 - d_1) = \deg d_2 \text{ or } \deg(d_2 - d_1) < \deg d_1 - 1, \\
\deg v_2 + \deg d_1 + \tau & \text{if } \deg(d_2 - d_1) = \deg d_1 - 1,
\end{cases}
$$

where in the last case $\tau$ is equal to $\text{lcm}(d_2 - d_1)/\text{lcm}d_1$ if this is a nonnegative integer, and $-1$ otherwise.
Observe that if \( d_1 = d_2 = 1 \), then by this theorem \( \deg P < \deg v_2 \). This is in agreement with the properties of the decomposition of indefinite sums of rational functions.

Note the following. If the pair \( d_1/d_2, v_1/v_2 \) has properties Pa and Pb, then the rational function \( P/v_2 \) is irreducible (i.e., \( P \perp v_2 \)) by Theorem 3. But we can try to reduce \( P/d_2 \). Suppose this gives \( P'/d'_2 \), \( d_2 = d_2 d'_2 \). Then

\[
\frac{P(n)}{v_2(n)d_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)} = \frac{P'(n)}{v_2(n)d_2(n)} \prod_{k=n_0}^{n-1} \frac{d_1(k)}{d_2(k)d_2(k+1)}.
\]

5 Examples

With algorithm hydecomp we can get the following decompositions:

\[
\left( \frac{1}{n+1} - \frac{1}{n} \right) \prod_{k=0}^{n-1} \frac{1}{k+2} = (E - 1) \left( \frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2} \right) + \prod_{k=0}^{n-1} \frac{1}{k+2}; \quad (28)
\]

\[
\left( \frac{1}{n+1} - \frac{1}{n} - 1 \right) \prod_{k=0}^{n-1} \frac{1}{k+2} = (E - 1) \left( \frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2} \right) ; \quad (29)
\]

\[
\left( \frac{1}{n+1} - \frac{2}{n} - 2 \right) \prod_{k=0}^{n-1} \frac{1}{k+2} = (E - 1) \left( \frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2} \right) - \frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}. \quad (30)
\]

Using the approach from section 4 we can rewrite the term

\[
\frac{n+1}{n} \prod_{k=0}^{n-1} \frac{1}{k+2}
\]

in the right-hand sides of (28), (29), (30) as

\[
\frac{1}{n} \prod_{k=0}^{n-1} \frac{1}{k+1}.
\]

References


