CUBES POLYNOMIAL AND ITS DERIVATIVES

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Abstract

Let $\alpha_i(G)$ be the number of induced $i$-cubes of a graph $G$. Then the cubes polynomial $c(G, x)$ of $G$ is introduced as $\sum_{i \geq 0} \alpha_i(G)x^i$. It is shown that any function $f$ with two related, natural properties, is up to the factor $f(K_1, x)$ the cubes polynomial. The derivation $\partial G$ of a median graph $G$ is also introduced and it is proved that the cubes polynomial is the only function $f$ with the property $f'(G, x) = f(\partial G, x)$ provided that $f(G, 0) = |V(G)|$. Several relations that generalize many previous results for median graphs are also given. For instance, for any $s \geq 0$ we have $c^{(s)}(G, x + 1) = \sum_{i \geq 0} \frac{c^{(s)}(G, x)}{(i-s)!}$. 

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1 Introduction

Several graph polynomials have been introduced in the literature, see the book [3]. In this paper we introduce the cubes polynomial of a graph $G$ as $c(G, x) = \sum_{i \geq 0} \alpha_i(G)x^i$, where $\alpha_i(G)$ denotes the number of induced $i$-cubes of $G$.

We first show that the cubes polynomial has three nice properties: amalgamation, product, and expansion property. In fact, any function $f$ with the amalgamation and the expansion property is up to the factor $f(K_1, x)$ the cubes polynomial. Weaker conditions suffice to reach similar conclusions if we restrict to the class of median graphs. Then we introduce the derivation graph $\partial G$ of a median graph $G$ and prove that the cubes polynomial is the only function $f$ with the property $f(G, x) = f(\partial G, x)$ provided that its value in $x = 0$ equals the number of vertices. In the last section we prove several relations for median graphs involving the cubes polynomial, for instance, for any $s \geq 0$ we have

$$c^{(s)}(G, x) = \sum_{i \geq s} \frac{(-1)^{i-s}}{(i-s)!} c^{(i)}(G, x + 1).$$

These relations (widely) generalize some previous results [7, 12, 13].

The Cartesian product $G \Box H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ in which the vertex $(a, x)$ is adjacent to the vertex $(b, y)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of $k$ copies of $K_2$ is a hypercube or $k$-cube $Q_k$. Isometric subgraphs of hypercubes are called partial cubes. A graph $G$ is a median graph if there exists a unique vertex $x$ to every triple of vertices $u, v,$ and $w$ such that $x$ lies simultaneously on a shortest $u, v$-path, a shortest $u, w$-path, and a shortest $w, v$-path. Median graphs are partial cubes, cf. [10, 4]. For more information on median graphs see [4, 8, 9, 10].

Two edges $e = xy$ and $f = uv$ of $G$ are in the Djoković-Winkler [2, 15] relation $\Theta$ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. Winkler [15] proved that a bipartite graph is a partial cube if and only if $\Theta$ is transitive.

Let $\mathcal{G}$ denote the set of all finite graphs, and $\mathcal{M}$ the class of all median graphs. Finally, Let $A_1, \ldots, A_n$ be sets and let $\mathcal{I} = \{1, \ldots, n\}$. Then the inclusion-exclusion property for sets $A_1, \ldots, A_n$ says:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{J \subseteq \mathcal{I}} (-1)^{|J|-1} \left| \bigcap_{i \in J} A_i \right|. \quad (1)$$
The cubes polynomial

For a graph $G$, let $\alpha_i(G)$ denote the number of induced $i$-cubes of $G$. Note that $\alpha_0(G) = |V(G)|$ and $\alpha_1(G) = |E(G)|$. Let $c(G, x)$ be the generating function of the sequence $(\alpha_i(G))_{i \geq 0}$, that is,
$$c(G, x) = \sum_{i \geq 0} \alpha_i(G)x^i.$$ 

For finite graphs $G$, $c(G, x)$ is a polynomial and we call it the cubes polynomial of $G$. For instance, let $T$ be a tree on $n$ vertices, then $c(T, x) = (n - 1)x + n$. Note also that $c(Q_n, x) = (x + 2)^n$ and that 
$$\alpha_k(G) = \frac{c^k(0)}{k!}.$$ 

A cover $C$ of a graph $G$ is a set of induced subgraphs $C = \{G_1, \ldots, G_n\}$ of $G$ such that $G = G_1 \cup G_2 \cup \cdots \cup G_n$. We say that a cover $C$ is proper, if every induced hypercube of $G$ is contained in at least one of the graphs of $C$. Let $I_n = \{1, \ldots, n\}$. For any subset $A \subseteq I_n$, let $G_A$ be the intersection (possibly empty and possibly disconnected) of the graphs $G_i$ ($i \in I$).

**Proposition 1** Let $C = \{G_1, \ldots, G_n\}$ be a proper cover of a graph $G$. Then 
$$c(G, x) = \sum_{A \subseteq I_n} (-1)^{|A|-1} c(G_A, x).$$ 

**Proof.** For $i \geq 0$ and $j \in \{1, 2, \ldots, n\}$, let $A_j^i$ be the set of induced $i$-cubes of the graph $G_j$. Then $\alpha_i(G) = |A_1^i \cup A_2^i \cup \cdots \cup A_n^i|$. Hence, by (1), we infer 
$$\alpha_i(G) = \sum_{A \subseteq I_n} (-1)^{|A|-1} |\bigcap_{j \in A} A_j^i|.$$ 

In addition, $c(G_A, x) = \sum_{i \geq 0} |\bigcap_{j \in A} A_j^i| x^i$. Therefore,
$$c(G, x) = \sum_{i \geq 0} \alpha_i(G)x^i$$ 
$$= \sum_{i \geq 0} \left( \sum_{A \subseteq I_n} (-1)^{|A|-1} |\bigcap_{j \in A} A_j^i| \right)x^i$$ 
$$= \sum_{A \subseteq I_n} \sum_{i \geq 0} \left( (-1)^{|A|-1} |\bigcap_{j \in A} A_j^i| \right)x^i$$ 
$$= \sum_{A \subseteq I_n} (-1)^{|A|-1} c(G_A, x).$$ 

3
Because of Proposition 1 we say that a function \( f : \mathcal{G} \times \mathbb{R} \to \mathbb{R} \) has the \textit{amalgamation property} if

\[
 f(G, x) = \sum_{A \subseteq I_n} (-1)^{|A| - 1} f(G_A, x).
\]  
whenever \( \{G_1, \ldots, G_n\} \) is a proper cover of \( G \).

As hypercubes are the simplest Cartesian product graphs, the cubes polynomial should behave nicely with respect to the Cartesian product. Indeed, observe that an induced \( r \)-cube of \( G \square H \) is uniquely representable as \( Q_r \square Q_r \), where \( Q_r \) is an induced \( s \)-cube of \( G \) and \( Q_r \), an induced \( (r - s) \)-cube of \( H \). Hence, for every \( k \geq 0 \),

\[
 \alpha_k(G \square H) = \sum_{i=0}^{k} \alpha_i(G) \alpha_{k-i}(H).
\]

From here we easily conclude:

**Proposition 2** For any graphs \( G \) and \( H \),

\[
 c(G \square H, x) = c(G, x) c(H, x).
\]

Observe that the identity \( |E(G \square H)| = |E(G)||V(H)| + |E(H)||V(G)| \) immediately follows from Proposition 2. We say that a function \( f : \mathcal{G} \times \mathbb{R} \to \mathbb{R} \) has the \textit{product property} if for any graphs \( G \) and \( H \),

\[
 f(G \square H, x) = f(G, x) f(H, x).
\]

Let \( G \) be a connected graph. The \textit{expansion} \( G^s \) of \( G \) with respect to the proper cover \( \{G_1, G_2\} \) of \( G \) is the graph constructed as follows. Let \( G_i^s \) be an isomorphic copy of \( G_i \), for \( i = 1, 2 \), and, for any vertex \( u \) of \( G_0 = G_1 \cap G_2 \), let \( u_i \) be the corresponding vertex in \( G_i^s \). Then \( G^s \) is obtained from the disjoint union of \( G_1^s \) and \( G_2^s \), where for each \( u \) of \( G_0 \) the vertices \( u_1 \) and \( u_2 \) are joined by an edge. (Note that, since the cover is proper, a vertex of \( G_1 \setminus G_2 \) cannot be adjacent to a vertex of \( G_2 \setminus G_1 \).)

It is easy to prove the following claim. Just observe that via the expansion the subgraph isomorphic to \( G_0 \square K_2 \) gives rise to new/larger hypercubes.

**Proposition 3** Let \( G^s \) be a graph constructed by the expansion with respect to the proper cover \( \{G_1, G_2\} \) (over \( G_0 \)). Then \( c(G^s, x) = c(G_1, x) + c(G_2, x) + x c(G_0, x) \).
Because of this result, the following definition seems reasonable. A function $f : G \times \mathbb{R} \to \mathbb{R}$ has the expansion property if

$$f(G^*, x) = f(G_1, x) + f(G_2, x) + x f(G_0, x),$$

whenever $G^*$ is the expansion with respect to the proper cover $\{G_1, G_2\}$ (over $G_0$).

**Theorem 4** Let $f : G \times \mathbb{R} \to \mathbb{R}$ be a function with the amalgamation and the expansion property. Then for any graph $G$,

$$f(G, x) = f(Q_0, x) c(G, x).$$

**Proof.** The proof is by induction on the number of vertices of a graph. The $n$-cube $Q_n$, $n \geq 1$, can be obtained as the expansion of the proper cover $\{Q_{n-1}, Q_{n-1}\}$ (over $Q_{n-1}$). Now, by the expansion property we have $f(Q_n, x) = (2 + x)f(Q_{n-1}, x)$. Hence,

$$f(Q_n, x) = (2 + x)^n f(Q_0, x) = c(Q_n, x) f(Q_0, x).$$

Suppose now that $G$ is a graph different from hypercubes. Let $C$ be the set comprised of graphs $G - v$ for every $v \in V(G)$. Since $G$ is not a hypercube, $C$ is a proper cover and every graph of $C$ is smaller than $G$. Using amalgamation property for $c$, and the induction hypothesis for graphs $G_i$, we infer that

$$f(G, x) = \sum_{A \subseteq I_n} (-1)^{|A|-1} f(G_A, x)$$

$$= \sum_{A \subseteq I_n} (-1)^{|A|-1} f(Q_0, x) c(G_A, x)$$

$$= f(Q_0, x) c(G, x).$$

From the above proof we can also deduce that if $f$ has the amalgamation property and $f(Q_n, x) = (x + 2)^n$ holds for every $n \in \mathbb{N}_0$, then $f \equiv c$.

In the case of median graphs we can further strengthen the result of Theorem 4. Recall that Mulder [9, 10] proved that a graph is a median graph if and only if it can be obtained from $K_1$ by a sequence of expansions in which $G_0 = G_1 \cap G_2$ is always convex, cf. also [8, 11].
Corollary 5  Let $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a function with the expansion property. Then $f(G, x) = f(Q_0, x)c(G, x)$. If, in addition, $f$ has the product property, then either $f \equiv 0$ or $f \equiv c$.

Proof. Let $f$ has the expansion property. The proof of the first assertion is by induction on the number of expansion steps needed to obtain a median graph. The conclusion is clear for $G = K_1$. Let now $G'$ be a median graph obtained by a (convex) expansion with respect to the proper cover $\{G_1, G_2\}$ (over $G_0$). Then, $f(G', x) = f(G_1, x) + f(G_2, x) + x f(G_0, x)$. From the induction hypothesis we obtain

$$f(G', x) = f(Q_0, x)c(G_1, x) + f(Q_0, x)c(G_2, x) + x f(Q_0, x)c(G_0, x),$$

and so $f(G', x) = f(Q_0, x)(c(G_1, x) + c(G_2, x) + x c(G_0, x)) = f(Q_0, x)c(G')$.

If $f$ also has the product property, then, since $Q_0 \sqcap Q_0 = Q_0$, we have $f(Q_0, x) = [f(Q_0, x)]^2$. Thus either $f(Q_0, x) = 0$ or $f(Q_0, x) = 1$. Combining this with the the fact that $f(G, x) = f(Q_0, x)c(G, x)$ completes the proof.

3 Derivation graphs of median graphs

Let $F$ be a $\Theta$-class of a median graph and let $e = uv \in F$. Then it is well-known that $F$ forms a matching. Moreover, the endvertices of edges of $F$ that are closer to $u$ than to $v$ induce a median graph isomorphic to the subgraph induced by the endvertices of edges of $F$ that are closer to $v$ than to $u$. We denote this median subgraph by $U_e$. Let $\mathcal{F}(G)$ be the set of edges consisting of representatives of the $\Theta$-classes of $G$. Then we define the derivation of a median graph $G$ as the graph

$$\partial G = \bigcup_{e \in \mathcal{F}(G)} U_e,$$

that is, as the disjoint union of the sets $U_e$, $e \in \mathcal{F}(G)$. (Note that transitivity of $\Theta$ implies that the graph $\partial G$ is well-defined.) For instance, $\partial (P_n \sqcap P_m)$ is the disjoint union of $n - 1$ copies of $P_m$ and $m - 1$ copies of $P_n$.

The reason for calling the graph $\partial G$ “the derivation” of $G$ is the following property.

Proposition 6 (Derivation property) Let $G$ be a median graph. Then,

$$c'(G, x) = c(\partial G, x).$$

(5)
Note that for any $Q_n$ in $G$ its edges lie in $n$ $\Theta$-classes ($n \geq 1$), and the corresponding graphs $U_e$ induce $(n-1)$-cubes—altogether there are $n$ $Q_{n-1}$’s for each $n$-cube of $G$. Hence, $\alpha_{n-1}(G)$ equals the number of $(n-1)$-cubes of $\partial G$. In other words:

$$c'(G, x) = \sum_{e \in F} c(U_e, x) = c(\partial G, x).$$

We say that a function $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ has the derivation property if

$$f'(G, x) = f(\partial G, x).$$

We next prove that the cubes polynomial is the only function on median graphs with the derivation property such that its value in $x = 0$ equals the number of vertices.

**Theorem 7** Let $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a function with the derivation property, such that $f(G, 0) = |V(G)|$. Then $f \equiv c$.

**Proof.** First, since $Q_0$ has no $\Theta$-classes, $\partial Q_0$ is empty. Thus $f(Q_0, x)$ is a constant, and $f(Q_0, 0) = 1$ implies $f(Q_0, x) = 1$. Hence $f(Q_0, x) = c(Q_0, x)$.

The proof proceeds by induction on the number of vertices of a median graph. Suppose that for any median graph $H$ with less than $k$ vertices we have $f(H, x) = c(H, x)$, and let $G$ has $k$ vertices. Recall that for every $e \in E(G)$, the graph $U_e$ is a median graph. Since $U_e$ has less than $k$ vertices, $f(U_e, x) = c(U_e, x)$ holds by the induction assumptions. By the derivation properties of $c$ and $f$ we infer that $f'(G, x) = c'(G, x)$. Thereby $f(G, x) = c(G, x) + C$, where $C$ is a constant. Since $f(Q_0, 0) = c(Q_0, 0)$, we derive that $C = 0$, thereof $f \equiv c$. □

Let us denote by $\mathcal{M}^*$ the class of all graphs whose connected components are median graphs. Thus, $G \in \mathcal{M}^*$ can be written as $G = G_1 \cup G_2 \cup \cdots \cup G_s$, where every $G_i$ is a median graph. Then we can extend the concept of the derivation graph to the graphs from $\mathcal{M}^*$ by setting $\partial G = \partial G_1 \cup \partial G_2 \cup \cdots \cup \partial G_s$ and also define the higher derivations in the following way. For $k \geq 0$, set

$$\partial^k G = \begin{cases} \partial (\partial^{k-1} G) & k \geq 1 \\ G & k = 0. \end{cases}$$

(6)

Then one can easily extend Proposition 6 to the graphs from $\mathcal{M}^*$ as well as to generalize it to the higher derivatives in the following way:

$$c^{(k)}(G, x) = c(\partial^k G, x).$$

(7)
4 Relations for median graphs

Throughout this section let $G$ be a median graph with $k$ $\Theta$-classes. The following two relations are known:

$$
\sum_{i \geq 0} (-1)^i \alpha_i(G) = 1 \quad \text{and} \quad k = -\sum_{i \geq 0} (-1)^i i \alpha_i(G). \tag{8}
$$

The first of these relations is due to Soltan and Chepoi [13, Theorem 4.2 (6)].

It was later independently obtained by Škrekovski in [12], where the second relation is also proved. Note that the first equality presents a generalization of the well-known equality “$n - m = 1$” for trees, while the second one applied to trees says that “$k = m$”, which is another characterizing property of trees. These relations also imply the Euler-type formulas from [7, 6].

Denote by the $s$ ($s \geq 0$) the number of components in the graph $\partial^s G$. Thus, $\theta_0(G) = 1$ and $\theta_1(G) = k$. Then we can extend (8) as follows.

(Recall that a graph is a median graph if and only if it can be obtained from hypercubes by a sequence of convex amalgams, a result due to Bandelt and van de Vel [1], cf. also [8, 14].)

**Proposition 8** Let $G$ be a median graph and $s \geq 0$. Then

$$
\theta_s(G) = c^{(s)}(G, -1).
$$

**Proof.** The proof is by induction. So, let first $s = 0$. If $G \cong Q_n$ ($n \geq 0$) then $c(Q_n, -1) = 1 = \theta_0(Q_n)$. Now assume that $G$ is the amalgam of $G_1$ and $G_2$ over $G_0$. Then, by the induction assumption,

$$
c(G, -1) = c(G_1, -1) + c(G_2, -1) - c(G_0, -1) = 1 + 1 - 1 = 1 = \theta_0(G).
$$

Suppose now that the claim holds for all integers smaller that $s$ ($s \geq 1$) and for all median graphs with less vertices than $G$. Since $c^{(s+1)}(G, x) = \sum_{e \in F(G)} c^{(s)}(U_e, x)$ we can use the induction hypothesis for graphs $U_e$ and derive

$$
c^{(s+1)}(G, -1) = \sum_{e \in F(G)} \theta_s(U_e) = \theta_{s+1}(G).
$$

\[\square\]
Theorem 9 Let $G$ be a median graph and $s \geq 0$. Then,

$$c^{(s)}(G, x + 1) = \sum_{i \geq s} c^{(i)}(G, x) \frac{e^{(i)}(G, x)}{(i - s)!}, \quad (9)$$

$$c^{(s)}(G, x) = \sum_{i \geq s} \frac{(-1)^{i-s}}{(i - s)!} c^{(i)}(G, x + 1). \quad (10)$$

Proof. The proof of the first equality is by induction on the number of amalgamation steps. Suppose first that $G \cong Q_n$. Since $c(Q_n, x) = (x + 2)^n$ it follows that $c^{(s)}(Q_n, x + 1) = \frac{n!}{(n-s)!} (x + 3)^{n-s}$. Using binomial formula we obtain:

$$c^{(s)}(Q_n, x + 1) = \frac{n!}{(n-s)!} \sum_{j=0}^{n-s} \binom{n-s}{j} (x + 2)^j$$

$$= \sum_{j=0}^{n-s} \frac{n!}{(n-s)!} \frac{(n-s)!}{j! (n-s-j)!} (x + 2)^j$$

$$= \sum_{j=0}^{n-s} \frac{c^{(s+j)}(Q_n, x)}{j!}$$

$$= \sum_{i \geq s} \frac{c^{(i)}(Q_n, x)}{(i - s)!},$$

and so the desired formula follows. If $G$ is not a hypercube then it can be obtained by an amalgamation of $G_1$ and $G_2$ over $G_0$. By the induction hypothesis we deduce:

$$c^{(s)}(G, x + 1) = c^{(s)}(G_1, x + 1) + c^{(s)}(G_2, x + 1) - c^{(s)}(G_0, x + 1)$$

$$= \sum_{i \geq s} \frac{1}{(i - s)!} \left( c^{(i)}(G_1, x) + c^{(i)}(G_2, x) - c^{(i)}(G_0, x) \right)$$

$$= \sum_{i \geq s} \frac{c^{(i)}(G, x)}{(i - s)!}.$$  

This proves the first relation.

The second equality can be proved in a similar way. Alternatively, one can write down the first equality for every $0 \leq s \leq r$, where $r$ is the dimension of a largest hypercube of $G$, and invert the obtained system of equations.  

$\square$
Corollary 10 Let $G$ be a median graph, $\alpha_i = \alpha_i(G)$ the number of induced $i$-cubes in $G$, and $\theta_i = \theta_i(G)$ the number of components in $\partial^i G$. Then for any $s \in \mathbb{N}_0$,

\begin{align*}
&\quad (a) \quad \alpha_s = \frac{1}{s!} \sum_{i \geq s} \theta_i \frac{1}{(i-s)!} \quad \text{and} \quad \theta_s = s! \sum_{i \geq 0} (-1)^{i-s} \binom{i}{s} \alpha_i.

&\quad (b) \quad \sum_{i \geq 0} (-1)^i 2^i \alpha_i = \sum_{i \geq 0} (-1)^i \frac{\theta_i}{i!}.
\end{align*}

**Proof.** In order to prove the first relation of (a), we set $x = -1$ in relation (10) to get

\[ (-1)^s c^{(s)}(G, -1) = \sum_{i \geq s} (-1)^i \frac{c^{(i)}(G, 0)}{(i-s)!}. \]

Therefore

\[ (-1)^s \frac{c^{(s)}(G, -1)}{s!} = \sum_{i \geq s} (-1)^i \frac{c^{(i)}(G, 0)}{s!(i-s)!}. \]

Now, using Proposition 8 and formula (2), we obtain

\[ (-1)^s \frac{\theta_s}{s!} = \sum_{i \geq s} (-1)^i \binom{i}{s} \alpha_i. \]

Finally note that $\binom{i}{s} = 0$ if $i < s$.

The proof of the second formula is obtained analogously as above by setting $x = -1$ in relation (9).

In order to obtain the relation (b) just sum up equalities from the first formula of (a) and use a basic property of binomial coefficients. \(\Box\)

Note that relations from (8) can be obtained from the second formula of Corollary 10(a) by setting $s = 0$ and $s = 1$, respectively.

**References**


