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$\bar{\partial}$-EQUATIONS IN THIN TUBES
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SOLVING THE $d$ AND $\bar{\partial}$-EQUATIONS IN THIN TUBES AND
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Abstract. We construct a family of integral kernels for solving the $\bar{\partial}$-equation with $C^k$ and Hörmander estimates in thin tubes around totally real submanifolds in $\mathbb{C}^n$ (theorems 1.1 and 3.1). Combining this with the proof of a theorem of Serre we solve the $d$-equation with estimates for holomorphic forms in such tubes (theorem 5.1). We apply these techniques and a method of Moser to approximate $C^k$-diffeomorphisms between totally real submanifolds in $\mathbb{C}^n$ in the $C^k$-topology by biholomorphic mappings in tubes, by unimodular and symplectic biholomorphic mappings, and by automorphisms of $\mathbb{C}^n$.

&1. The results.

Let $\mathbb{C}^n$ denote the complex $n$-dimensional Euclidean space with complex coordinates $z = (z_1, \ldots, z_n)$. We shall consider compact $C^k$-submanifolds $M \subset \mathbb{C}^n$ ($k \geq 1$), with or without boundary. Such a submanifold is totally real in $\mathbb{C}^n$ if for each $z \in M$ the tangent space $T_z M$ (which is a real subspace of $T_z \mathbb{C}^n$) contains no complex line; equivalently, the complex subspace $T_z^C M = T_z M + iT_z M$ of $T_z \mathbb{C}^n$ has complex dimension $m = \dim \mathbb{R} M$ for each $z \in M$. We let $\mathcal{T}_\delta M = \{ z \in \mathbb{C}^n; d_M(z) < \delta \}$ denote the tube of radius $\delta > 0$ around $M$; here $|z|$ is the Euclidean length of $z$ and $d_M(z) = \inf \{|z - w|; w \in M\}$.

For any open set $U \subset \mathbb{C}^n$ and integers $p,q \in \mathbb{Z}_+$ we denote by $C^1_{p,q}(U)$ the space of differential forms of class $C^1$ and of bidegree $(p,q)$ on $U$. For each multiindex $\alpha \in \mathbb{Z}_+^m$ we denote by $\partial^\alpha$ the corresponding partial derivative of order $|\alpha|$ with respect to the underlying real coordinates on $\mathbb{C}^n$.

The following is one of the main results of the paper; for additional estimates see theorem 3.1 in sect. 3 below.

1.1 Theorem. Let $M \subset \mathbb{C}^n$ be a closed, totally real, $C^1$-submanifold and let $0 < c < 1$. Denote by $\mathcal{T}_\delta$ the tube of radius $\delta > 0$ around $M$. There exist constants $\delta_0 > 0$ and $C > 0$ such that the following holds for all $0 < \delta \leq \delta_0$, $l \geq 1$, $p \geq 0$, $q \geq 1$: For any $u \in C^1_{p,q}(\mathcal{T}_\delta)$ with $\bar{\partial} u = 0$ there is a $v \in C^1_{p,q-1}(\mathcal{T}_\delta)$ satisfying $\bar{\partial} v = u$ in $\mathcal{T}_{c\delta}$ and the estimates

$$
\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \leq C\delta \|u\|_{L^\infty(\mathcal{T}_\delta)};
$$

$$
\|\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \leq C \left( \delta \|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{1-|\alpha|} \|u\|_{L^\infty(\mathcal{T}_\delta)} \right); \quad |\alpha| \leq l.
$$

If $q = 1$ and the equation $\bar{\partial} v = u$ has a solution $v_0 \in C^1_{(p,0)}(\mathcal{T}_\delta)$, there is a solution $v \in C^1_{(p,0)}(\mathcal{T}_\delta)$ of $\bar{\partial} v = u$ satisfying for $1 \leq j \leq n$ and $|\alpha| = l$:

$$
\|\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \leq C \left( \omega(\partial^\alpha v_0, \delta) + \delta^{-l} \|v_0\|_{L^\infty(\mathcal{T}_\delta)} \right).
$$
Here $\omega(f,\delta) = \sup \{|f(x)-f(y)|; |x-y| \leq \delta\}$ is the modulus of continuity of a function; when $f$ is a differential form on $\mathbb{C}^n$, $\omega(f,t)$ is defined as the sum of the moduli of continuity of its components (in the standard basis).

The solution in theorem 1.1 is obtained by a family of integral kernels, depending on $\delta > 0$ and constructed specifically for thin tubes (and hence is given by a linear solution operator on each tube $\mathcal{T}_{\delta}$). Immediate examples show that the gain of $\delta$ in the estimate for $v$ is the best possible. When $u$ is a $(0,1)$-form (or a $(p,1)$-form), the estimates for the derivatives of $v$ in (1.1) follow from the sup-norm estimate by shrinking the tube and applying the interior regularity for the $\bar{\partial}$-operator (lemma 3.2). This is not the case in bidegrees $(p,q)$ for $q > 1$. We refer to section 3 below for further details.

Another major result of the paper is theorem 5.1 in section 5 on solving the equation $dv = u$ for holomorphic forms in tubes $\mathcal{T}_{\delta}$ with precise estimates. Theorem 5.1 is obtained by using the solutions of the $\bar{\partial}$-equation, provided by theorem 3.1, in the proof of Serre’s theorem to the effect that, on pseudoconvex domains, the de Rham cohomology groups are given by holomorphic forms.

We now apply these results to the problem of approximating smooth diffeomorphisms between totally real submanifolds in $\mathbb{C}^n$ by biholomorphic maps in tubes $\mathcal{T}_{\delta}$ and by holomorphic automorphism of $\mathbb{C}^n$. The technical tools developed here are very precise and give optimal results without any loss of derivatives in these approximation problems.

Recall that the complex normal bundle $\nu_M \to M$ of a totally real submanifold $M \subset \mathbb{C}^n$, defined as the quotient bundle $\nu_M = TC^0|_M/T^C M$, can be realized as a complex subbundle of $TC^0|_M$ such that $TC^0|_M = T^C M \oplus \nu_M$. Given a diffeomorphisms $f: M_0 \to M_1$ between totally real submanifolds $M_0, M_1 \subset \mathbb{C}^n$, we say that the complex normal bundles $\pi_i: \nu_i \to M_i$ are isomorphic over $f$ if there exists a $C$-vector bundle map $\phi: \nu_0 \to \nu_1$ satisfying $\pi_1 \circ \phi = f \circ \pi_0$.

**1.2 Theorem.** Let $f: M_0 \to M_1$ be a diffeomorphism of class $C^k$ between compact totally real submanifolds $M_0, M_1 \subset \mathbb{C}^n$, with or without boundary ($n \geq 1, k \geq 2$). Assume that the complex normal bundles to $M_0$ and $M_1$ are isomorphic over $f$. Then there are numbers $\delta_0 > 0$ and $a > 0$ such that for each $\delta \in (0,\delta_0)$ there exists an injective holomorphic map $F_\delta: \mathcal{T}_{\delta} M_0 \to \mathbb{C}^n$ such that $F_\delta(\mathcal{T}_{\delta} M_0) \supset \mathcal{T}_{a\delta} M_1$ and the following estimates hold for $0 \leq r \leq k$ as $\delta \to 0$:

$$
\|F_\delta|_{M_0} - f\|_{C^r(M_0)} = o(\delta^{k-r}), \quad \|F_\delta^{-1}|_{M_1} - f^{-1}\|_{C^r(M_1)} = o(\delta^{k-r}). \quad (1.2)
$$

The $C^r(M)$-norm is defined as usual by using a finite open covering of $M$ by coordinate charts and a corresponding partition of unity; see sect. 3 for the details. An important aspect of theorem 1.2 is the precise relationship between the rate of approximation on $M_0$ resp. $M_1$ and the radius $\delta$ of the tube on which the approximating biholomorphic map $F_\delta$ is defined. The condition on the isomorphism of the complex normal bundles over $f$ is a necessary one since any biholomorphic map defined near $M_0$ which is sufficiently close to $f$ in the $C^1(M_0)$-norm induces such an isomorphism. If $M_0$ and $M_1$ are contractible (such as
arcs or totally real discs), or if they are of maximal real dimension $n$, theorem 1.2 applies to any $C^k$-diffeomorphism $f: M_0 \to M_1$.

When all data in theorem 1.2 are real-analytic, $f$ extends to a biholomorphic map $F$ from a neighborhood of $M_0$ onto a neighborhood of $M_1$ (see remark 1 after the proof of theorem 1.2 in sect. 4). In such case we say that $M_0$ and $M_1$ are biholomorphically equivalent; such pairs of submanifolds have identical local analytic properties in $\mathbb{C}^n$. This is not so if $f$ is smooth but non real-analytic, for there exist smooth arcs in $\mathbb{C}^n$ which are complete pluripolar as well as arcs which are not pluripolar [DF], yet any diffeomorphism between smooth arcs can be approximated as in theorem 1.2.

We don’t know whether in general there exist biholomorphic maps $F_\delta$ in a fixed open neighborhood of $M_0$ and satisfying (1.2) as $\delta \to 0$. However, in certain situations we can approximate diffeomorphisms by global holomorphic automorphisms of $\mathbb{C}^n$. Before stating the result we recall some relevant notions. A compact set $K \subset \mathbb{C}^n$ is polynomially convex if for each $z \in \mathbb{C}^n \setminus K$ there is a holomorphic polynomial $P$ on $\mathbb{C}^n$ such that $|P(z)| > \sup \{|P(x)| : x \in K\}$. Aut$\mathbb{C}^n$ denotes the group of all holomorphic automorphisms of $\mathbb{C}^n$.

**Definition 1.**

(a) A $C^k$-isotopy (or a $C^k$-flow) in $\mathbb{C}^n$ is a family of $C^k$-diffeomorphisms $f_t: M_0 \to M_t$ $(t \in [0,1])$ between $C^k$-submanifolds $M_t \subset \mathbb{C}^n$ such that $f_0$ is the identity on $M_0$, and both $f_t(z)$ and $\frac{\partial}{\partial t} f_t(z)$ are continuous with respect to $(t, z) \in [0,1] \times M_0$ and of class $C^k(M_0)$ in the second variable for each fixed $t \in [0,1]$.

(b) The isotopy in (a) is said to be totally real (resp. polynomially convex) if the submanifold $M_t \subset \mathbb{C}^n$ is totally real (resp. compact polynomially convex) for each $t \in [0,1]$.

(c) The infinitesimal generator of $f_t$ as in (a) is the time-dependent vector field $X_t$ on $\mathbb{C}^n$ which is uniquely defined along $M_t$ by the equation $\frac{\partial}{\partial t} f_t(z) = X_t(f_t(z))$ ($z \in M_0$, $t \in [0,1]$).

(d) A holomorphic isotopy (or holomorphic flow) on a domain $D \subset \mathbb{C}^n$ is a family of injective holomorphic maps $F_t: D \to \mathbb{C}^n$ such that $F_0$ is the identity on $D$ and such that the maps $F_t(z)$ and $\frac{\partial}{\partial t} F_t(z)$ are continuous with respect to $(t, z) \in [0,1] \times D$. Its infinitesimal generator $X_t$, defined as in (c), is a holomorphic vector field on the domain $D_t = F_t(D)$ for each $t \in [0,1]$.

**1.3 Theorem.** Let $M_0 \subset \mathbb{C}^n$ be a compact $C^k$-submanifold of $\mathbb{C}^n$ $(n \geq 2$, $k \geq 2$). Assume that $f_t: M_0 \to M_t \subset \mathbb{C}^n$ $(t \in [0,1])$ is a $C^k$-isotopy such that the submanifold $M_t = f_t(M_0) \subset \mathbb{C}^n$ is totally real and polynomially convex for each $t \in [0,1]$. Set $f = f_1: M_0 \to M_1$. Then there exists a sequence of holomorphic automorphisms $F_j \in \text{Aut}\mathbb{C}^n$ $(j = 1, 2, 3, \ldots)$ such that

$$\lim_{j \to \infty} \|F_j\|_{M_0} - f\|_{C^k(M_0)} = 0, \quad \lim_{j \to \infty} \|F_j^{-1}\|_{M_1} - f^{-1}\|_{C^k(M_1)} = 0. \quad (1.3)$$

Combining theorem 1.3 with corollary 4.2 from [FR] we obtain:
1.4 Corollary. Let \( f: M_0 \to M_1 \) be \( C^k \)-diffeomorphism \((k \geq 2)\) between compact, totally real, polynomially convex submanifolds of \( \mathbb{C}^n \) of real dimension \( m \). If \( 1 \leq m \leq 2n/3 \), there exists a sequence \( F_j \in \text{Aut}\mathbb{C}^n \) \((j = 1, 2, 3, \ldots)\) satisfying \((1.3)\).

Theorems 1.2 and 1.3 are proved in sect. 4 below. A weaker version of theorem 1.3 (with loss of derivatives) was obtained in [FL] by applying Hörmander’s \( L^2 \)-method for solving the \( \bar{\partial} \)-equations in tubes. For a converse to theorem 1.3 see remark 2 on p. 135 in [FL]. When \( f \) is a real-analytic diffeomorphism as in theorem 1.3, the approximating sequence of automorphisms \( F_j \in \text{Aut}\mathbb{C}^n \) can be chosen such that it converges to a biholomorphic map \( F \) in an open neighborhood of \( M_0 \) in \( \mathbb{C}^n \) satisfying \( F|_{M_0} = f \) [FR].

We now consider the approximation problem for maps preserving one of the forms

\[
\omega = dz_1 \wedge \cdots \wedge dz_n, \tag{1.4}
\]

\[
n = 2n', \quad \omega = \sum_{j=1}^{n'} dz_{2j-1} \wedge dz_{2j}. \tag{1.5}
\]

A holomorphic map \( F \) between domains in \( \mathbb{C}^n \) satisfying \( F^* \omega = \omega \) will be called a holomorphic \( \omega \)-map. \((1.4)\) is the (standard) complex volume form on \( \mathbb{C}^n \); in this case \( F^* \omega = JF \cdot \omega \), where \( JF \) is the complex Jacobian determinant of \( F \), and \( \omega \)-maps are called unimodular. \((1.5)\) is the standard holomorphic symplectic form, and holomorphic \( \omega \)-maps are called symplectic holomorphic. We denote the corresponding automorphism group by

\[
\text{Aut}_\omega \mathbb{C}^n = \{ F \in \text{Aut}\mathbb{C}^n : F^* \omega = \omega \}.
\]

For convenience we state the approximation results for \( \omega \)-maps (theorems 1.5, 1.7 and corollary 1.6) only for closed submanifolds; for an extension to manifolds with boundary see the remark following theorem 1.7.

1.5 Theorem. Let \( \omega \) be any of the forms \((1.4), (1.5)\). Let \( f: M_0 \to M_1 \) be a \( C^k \)-diffeomorphism between closed totally real submanifolds in \( \mathbb{C}^n \) \((k, n \geq 2)\). Assume that there is a \( C^{k-1} \)-map \( L: M_0 \to GL(n, \mathbb{C}) \) satisfying

\[
L_z|_{T_zM_0} = df_z, \quad L_z^* \omega = \omega \quad (z \in M_0). \tag{1.6}
\]

Then for each sufficiently small \( \delta > 0 \) there is an injective holomorphic map \( F_\delta: T_\delta M_0 \to \mathbb{C}^n \) such that \( F_\delta^* \omega = \omega \) and \((1.2)\) holds as \( \delta \to 0 \). If \( M_0, M_1 \) and \( f \) are real-analytic and if there exists a continuous \( L \) satisfying \((1.6)\), then \( f \) extends to a biholomorphic map \( F \) on a neighborhood of \( M_0 \) satisfying \( F^* \omega = \omega \).

The notation \( L_z^* \omega \) in \((1.6)\) denotes the pull-back of the multi-covector \( \omega_{f(z)} \) by the \( \mathbb{C} \)-linear map \( L_z \) (which we may interpret as a map \( T_z \mathbb{C}^n \to T_{f(z)} \mathbb{C}^n \)). Clearly \((1.6)\) implies that the complex normal bundles \( \nu_j \to M_j \) are isomorphic over \( f \). The condition in theorem 1.5 can be expressed as follows:

\(^*\) There exists a \( C^{k-1} \)-map \( L: M_0 \to SL(n, \mathbb{C}) \) (resp. \( L: M_0 \to SP(n, \mathbb{C}) \)) such that \( L_z = df_z \) on \( T_zM_0 \) for each \( z \in M_0 \).
Here $SL(n, \mathbb{C})$ (resp. $SP(n, \mathbb{C})$) is the linear unimodular (resp. linear symplectic) group on $\mathbb{C}^n$ (resp. on $\mathbb{C}^{2n}$).

The only obvious necessary condition for the approximation of a $C^k$-diffeomorphism $f: M_0 \to M_1$ by holomorphic $\omega$-maps is that the complex normal bundles $\nu_j \to M_j$ are isomorphic over $f$ and $f^*(i^*_j \omega) = i^*_0 \omega$, where $i_j: M_j \hookrightarrow \mathbb{C}^n$ is the inclusion. Theorem 1.5 reduces this (analytic) approximation problem to the geometric problem of finding an extension $L$ of $df$ satisfying (1.6). The regularity of $L$ is not the key point; in fact it would suffice to assume the existence of a continuous $L$ satisfying (1.6), since an argument similar to the one in the proof of theorem 1.5 for the real-analytic case then allows us to approximate $L$ by a $C^{k-1}$-map satisfying (1.6). We expect that such extension does not always exist, although we do not have specific examples at this time. Here are some positive results in this direction.

1.6 Corollary. Let $\omega$ be one of the forms (1.4), (1.5), and let $k, n \geq 2$. Let $f: M_0 \to M_1$ be a $C^k$-diffeomorphism between closed totally real submanifolds such that the complex normal bundles to $M_0$ resp. $M_1$ in $\mathbb{C}^n$ are isomorphic over $f$ and $f^* \omega = i^*_0 \omega$. Then the conclusion of theorem 1.5 holds in each of the following cases:

(i) $\dim M_0 = \dim M_1 = n$,
(ii) $\omega = dz_1 \wedge \cdots \wedge dz_n$ and $M_0$ is simply connected,
(iii) $\omega = dz_1 \wedge \cdots \wedge dz_n$ and $\nu_0$ admits a complex line subbundle.

In cases (ii) and (iii) we have $f^* \omega = i^*_0 \omega = 0$ when $m < n$. Finally we present approximation results for $\omega$-flows. We first introduce convenient terminology.

Definition 2. Let $\omega$ be a differential form on $\mathbb{C}^n$ and let $f_t: M_0 \to M_t \subset \mathbb{C}^n$ ($t \in [0, 1]$) be a $C^k$-isotopy with the infinitesimal generator $X_t$ (definition 1).

(a) $f_t$ is an $\omega$-flow if the form $f_t^* \omega$ on $M_0$ is independent of $t \in [0, 1]$.
(b) An $\omega$-flow $f_t$ is closed (resp. exact) if for each $t \in [0, 1]$ the pull-back to $M_t$ of the form $\alpha_t = X_t |_{M_t} \omega$ (the contraction of $\omega$ by $X_t$) is closed (resp. exact).
(c) Let $U \subset \mathbb{C}^n$ be an open set and $\omega$ a holomorphic form on $\mathbb{C}^n$. A holomorphic flow $F_t: U \to \mathbb{C}^n$ ($t \in [0, 1]$) satisfying $F_t^* \omega = \omega$ for all $t$ is called a holomorphic $\omega$-flow.

Remark. If $d\omega = 0$ (this holds for the forms (1.4), (1.5)) then a flow $f_t: M_0 \to M_t$ is an $\omega$-flow if and only if the pull-back of $\alpha_t = X_t |_{M_t} \omega$ to $M_t$ is a closed form on $M_t$ for each $t \in [0, 1]$. This can be seen from the following formula for the Lie derivative $L_{X_t} \omega$ ([AMR], p. 370, Theorem 5.4.1. and p.429, Theorem 6.4.8. (iv)):

$$\frac{d}{dt}(f_t^* \omega) = f_t^*(L_{X_t} \omega) = f_t^*(d(X_t |_{M_t}) \omega + X_t |_{M_t} d\omega) = f_t^*(d\alpha_t).$$

Hence $f_t^* \omega$ is independent of $t$ if and only if $d(i_t^* \alpha_t) = 0$ on $M_t$ for each $t \in [0, 1]$. 

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1.7 Theorem. Let $\omega$ be any of the forms (1.4), (1.5). Assume that $M_0 \subset \mathbb{C}^n$ is a closed totally real submanifold and $f_t: M_0 \to M_t \subset \mathbb{C}^n$ ($t \in [0, 1]$) is a totally real $\omega$-flow of class $C^k$ for some $k \geq 2$. Then for each sufficiently small $\delta > 0$ there is a holomorphic $\omega$-flow $F^\delta_t: T_0 M_0 \to \mathbb{C}^n$ ($t \in [0, 1]$) such that for $0 \leq r \leq k$ we have the following estimates as $\delta \to 0$ (uniformly with respect to $t \in [0, 1]$):

$$
\|F^\delta_t - f_t\|_{C^r(M_0)} = o(\delta^{k-r}), \quad \|(F^\delta_t)^{-1} - f_t^{-1}\|_{C^r(M_t)} = o(\delta^{k-r}).
$$

If in addition $n \geq 2$ and $f_t$ is an exact $\omega$-flow which is totally real and polynomially convex, there is for each $\epsilon > 0$ a holomorphic $\omega$-flow $F_t \in \text{Aut}_\omega \mathbb{C}^n$ such that for all $t \in [0, 1]$

$$
\|F_t - f_t\|_{C^k(M_0)} < \epsilon, \quad \|F_t^{-1} - f_t^{-1}\|_{C^k(M_t)} < \epsilon.
$$

Remark. Theorems 1.5, 1.7 and corollary 1.6 extend to the following situation. Let $M_0$ be a compact domain in a totally real submanifold $M'_0 \subset \mathbb{C}^n$, not necessarily closed or compact. In particular, $M_0$ may be a totally real submanifold with boundary $\partial M_0$ and $M'_0$ a larger submanifold containing $M_0$. In the context of theorem 1.5 or corollary 1.6 assume that $f: M'_0 \to M'_1$ is a $C^k$-diffeomorphism between totally real submanifolds in $\mathbb{C}^n$ ($k \geq 2$) and $L: M'_0 \to GL(n, \mathbb{C})$ is a $C^{k-1}$-map satisfying (1.6) on $M'_0$. Then the conclusion of theorem 1.5 holds for $M_0$: There exist holomorphic $\omega$-maps $F^\delta_t: \partial M_0 \to \mathbb{C}^n$ for all sufficiently small $\delta > 0$ satisfying (1.2) as $\delta \to 0$. Likewise, if the flow $f_t$ as in theorem 1.7 is defined on $M'_0$, the conclusion of that theorem applies on the compact subdomain $M_0 \subset M'_0$.

In our last result we consider the problem of approximating a diffeomorphism $f: M_0 \to M_1$ by holomorphic $\omega$-automorphisms of $\mathbb{C}^n$. Assuming that $M_0$ and $M_1$ are polynomially convex we have two necessary conditions for such approximation:

1. $f^* \omega = i_0^* \omega$, and
2. there is a totally real, polynomially convex flow $f_t: M_0 \to M_t \subset \mathbb{C}^n$ ($t \in [0, 1]$) with $f_0 = I d_{M_0}$ and $f_1 = f$.

The necessity of the second conditions follows from connectedness of the group $\text{Aut}_\omega \mathbb{C}^n$; see [FR]. When dim $M_0$ is smaller than the degree of $\omega$, the first condition is trivial since both sides are zero. We summarize some of the situations when such an approximation is possible. Let $\beta$ be a holomorphic form on $\mathbb{C}^n$ satisfying $d \beta = \omega$; when $\omega$ is given by (1.4) we may take $\beta = \frac{1}{n} \sum_{j=1}^n (-1)^{j-1} dz_1 \wedge \cdots \wedge \hat{dz}_j \cdots \wedge dz_n$, and when $\omega$ is the form (1.5) we may take $\beta = \sum_{j=1}^n z_{2j-1} dz_{2j}$.

1.8 Theorem. Let $n, k \geq 2$. Let $M_0 \subset \mathbb{C}^n$ be a compact connected $C^k$-submanifold of dimension $m$ and let $f_t: M_0 \to M_t$ ($t \in [0, 1]$) be a totally real, polynomially convex $C^k$-flow. Assume either that $\omega$ is the volume form (1.4), $d \beta = \omega$, and at least one of the following four conditions holds:
(i) $m \leq n - 2$;
(ii) $m = n - 1$ and $H^{n-1}(M_0; \mathbb{R}) = 0$;
(iii) $m = n - 1$, $M_0$ is closed and orientable, and $\int_{M_0} \beta = \int_{M_0} f_1^* \beta \neq 0$;
(iv) $m = n$, $M_0$ is closed and satisfies $H^{n-1}(M_0; \mathbb{R}) = 0$, and $f_1^* \omega$ is independent of $t$,
or that $n = 2n'$ ($n' \geq 2$), $\omega$ is the form $(1.5)$, $d\beta = \omega$, and at least one of the following
three conditions holds:
(v) $M_0$ is an arc;
(vi) $M_0$ is a circle and $\int_{M_0} \beta = \int_{M_0} f_1^* \beta$;
(vii) $m = 2$, $M_0$ is closed and satisfies $H^1(M_0; \mathbb{R}) = 0$, and $f_1^* \omega$ is independent of $t \in [0,1]$.
Set $f = f_1: M_0 \rightarrow M_1$. Then there is a sequence $F_j \in \text{Aut}_\omega \mathbb{C}^n$ satisfying $(1.3)$.

Remark. For real-analytic data the corresponding results were obtained in [F2] for the
symplectic case and in [F3] for the unimodular case. In that situation the approximating
sequence $F_j \in \text{Aut}_\omega \mathbb{C}^n$ can be chosen such that it converges to a holomorphic $\omega$-map $F$
in a neighborhood of $M_0$.

The paper is organized as follows. In sect. 2 we collect some preliminary material,
mostly extensions of certain well known results. In sect. 3 we construct a family of integral
kernels for solving the $\partial$-equation in tubes and we prove the stated estimates; we conclude
the section by historical remarks concerning such kernels. In sect. 4 we apply theorem 3.1
to prove theorems 1.2 and 1.3. In sect. 5 we solve the equation $dv = u$ in tubes, where
$u$ is an exact holomorphic form and we find a holomorphic solution $v$ satisfying good
estimates. In sections 6 and 7 we prove the results on approximating $\omega$-diffeomorphisms
by holomorphic $\omega$-maps and $\omega$-automorphisms. At the end of section 4 we also include a
correction to [FL].

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&2. Geometric preliminaries.

We denote by $X|v$ the contraction of a form $v$ by a vector field $X$. We shall use the
following version of the Poincaré’s lemma ([AMR], p.437, Deformation Lemma 6.4.17.):

2.1 Lemma. Let $M$ be a $C^2$-manifold and $u$ a closed $C^1$ $p$-form on $I \times M$; $I = [0,1]$, $p > 0$. For $t \in I$ let $i_t: M \rightarrow I \times M$ be the injection $x \rightarrow (t, x)$. Then the $(p - 1)$-form $v = \int_0^1 i_t^* \left( \frac{\partial}{\partial t} \right) |w) dt$ on $M$ satisfies $dv = i_1^* w - i_0^* w$. In particular, let $F: I \times M \rightarrow N$ be a
$C^2$-map and $u$ a closed $C^1$ $p$-form on $N$; $p > 0$. Setting $f_t = F \circ i_t: M \rightarrow N$ and $w = F^* u$ we get $dv = f_1^* u - f_0^* u$. 

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We shall apply this to the case when $F$ is a deformation retraction of a tubular neighborhood $T_\delta = T_\delta M$ of a submanifold $M \subset \mathbb{C}^n$ onto $M$. This means that $f_t$ is the identity on $T_\delta$, $f_t|_M$ is the identity for all $t$, and $f_0(T_\delta) = M$. Set $\pi = f_0$. With $u$ a closed $C^1$-form on $T_\delta$ and $v$ as above we get $dv = u - \pi^*u$ in $T_\delta$.

In the situation we shall consider we have the following local description of the retraction $F$. Let $M$ be a $C^k$-submanifold in $\mathbb{C}^n$. For $U$ a small open neighborhood in $M$ of a point $z_0 \in M$ there is a $C^k$-diffeomorphism $\phi: O \to \pi^{-1}(U)$, where $O$ is open in $\mathbb{R}^m \times \mathbb{R}^{2n-m}$, such that 

(i) $F^{-1}(U) = O \cap (\mathbb{R}^m \times \{0\}^{2n-m}) = O' \times \{0\}^{2n-m}$,

(ii) for $x' \in O'$, the set $O_{x'} = \{y' \in \mathbb{R}^{2n-m}: (x', y') \in O\}$ is starshaped with respect to 0,

(iii) the map $f_t = F \circ i_t$ is $\phi$-conjugate to $(x', y') \to (x', ty')$ for each $t \in I = [0, 1]$.

Let $u = \sum_{|I|+|J|=p} u_{I,J}(x',y')dx^I \wedge dy^J$ in these coordinates. Then 

$$w = \sum_{|I|+|J|=p} ' u_{I,J}(x',ty')dx^I \wedge d(ty')^J$$

and it is easy to check that 

$$v = \sum_{|I|+|K|=p} '( -1)^{|K|} \sum_{j=1}^n \sum_{|I|=|K|+1} \epsilon_{jK}^I y_j^I \left( \int_0^1 u_{I,J}(x',ty')y^I dt \right) dx^J \wedge dy^K,$$

where $\epsilon_{jK}^I$ equals, if $jK$ is a permutation of $J$, the signature of that permutation, and equals zero otherwise.

$F$ is constructed by retracting to $M$ along the fibers of a vector bundle supplementary to the tangent bundle $TM$. The normal bundle to $M$ in $\mathbb{C}^n$ is an obvious choice, but is only of class $C^{k-1}$ when $M$ is a $C^k$-submanifold. We shall show below that there are $C^k$-subbundles $E$ of $M \times \mathbb{C}^n$ that are arbitrarily close to the normal bundle. When $k > 1$, it is easy to see that $(z + E_z) \cap T_\delta$ is starshaped with respect to $z$ for all $z \in M$ when $\delta > 0$ is small enough and $E$ is sufficiently close to the normal bundle. The map $G: E \to \mathbb{C}^n$, $G(z,v) = z + v$, maps the zero section $0_E$ diffeomorphically onto $M$, and its derivative $dG$ is an isomorphism at each point of $0_E$; hence $G$ is a $C^k$-diffeomorphism of a neighborhood $U_\delta \subset E$ of $0_E$ onto $T_\delta$ for $\delta > 0$ small. We may assume that $U_\delta \cap E_z$ is starshaped with respect to $(z,0)$ for each $z \in M$. When $f_t$ is $G$-conjugate to the map $(z,v) \to (z, tv)$ in $U_\delta$ for $t \in I$, the map $F$ has the properties listed above.

The local coordinates $(x',y')$ are constructed as follows. Let $\varphi: O' \to U \subset M$ be a local $C^k$ parametrization, and $s_1, \ldots, s_{2n-m}$ sections of $E \to M$ over $U$ which form a $C^k$-trivialization of $E|_U$. We set $\phi(x',y') = \varphi(x') + \sum_{j=1}^{2n-m} y'_j s_j(\varphi(x'))$ for $x' \in O'$, $y' \in \mathbb{R}^{2n-m}$, and restrict it to $O = \varphi^{-1}(T_\delta)$. Then the fiber $O_x$ is starshaped for all $x' \in O'$ when $\delta > 0$ is small enough. ♦

2.2 Lemma. (Approximation of subbundles) Let $M$ be a $C^k$-submanifold of $\mathbb{C}^n$ and $E \to M$ a $C^l$-subbundle (real or complex) of $M \times \mathbb{C}^n$ for some $0 \leq l < k$. Then there is a
$C^k$-subbundle $E'$ of $M \times C^n$ arbitrarily close to $E$ in the $C^1$ topology. Moreover, if $M$ is totally real in $C^n$ and the bundle $E$ is complex, $E'$ may be taken as the restriction to $M$ of a holomorphic subbundle of $U \times C^n$ for some open neighborhood $U$ of $M$ in $C^n$.

**Proof.** A proof may be based on the following standard result. If $L: M \to \text{Lin}_C(C^n, C^n)$ is a $C^\ell$ map such that $L_z$ has constant rank $r$ independent of $z \in M$ (abusing the language we shall say that $L$ has rank $r$), then

$$E_L = \{(z, v) \in M \times C^n; v \in L_z(C^n)\}$$

is a complex $C^\ell$-subbundle of rank $r$ of the trivial bundle $M \times C^n$, and every subbundle $E$ of $M \times C^n$ appears in this manner, for instance by setting $L_z$ to be the orthogonal projection of $C^n$ onto the fiber $E_z$ for $z \in M$. The analogous result holds for real vector bundles.

A more regular approximation to a subbundle $E$ may then be obtained by approximating the corresponding map $L$ defining $E$ by a more regular map of rank $r$. The problem is that the rank of a generic perturbation of $L$ may increase. To overcome this we use the following result:

Let $C$ be a positively oriented simple closed curve in $C$, and let $L \in \text{Lin}_C(C^n, C^n)$ be a linear map with no eigenvalues on $C$. Then $C^n = V_+ \oplus V_-$, where $V_+$ resp. $V_-$ are $L$-invariant subspaces of $C^n$ spanned by the generalized eigenvectors of $L$ inside resp. outside of $C$. The map

$$P(L) = \frac{1}{2\pi i} \int_C (\zeta I - L)^{-1} \, d\zeta$$

(2.2)

is the projection onto $V_+$ with kernel $V_-$ (see [GLR]). Note that $P(L)$ depends holomorphically on $L$; thus, if $L$ depends $C^k$ or holomorphically on a parameter, so does $P(L)$.

We now take $C$ to be a curve which encircles 1 but not zero; for instance

$$C = \{\zeta \in C; |\zeta - 1| = 1/2\}.$$ 

(2.3)

Let $P$ be the associated projection operator (2.2). If $L$ is a projection then $P(L) = L$. Moreover, for each $L'$ sufficiently near a projection $L$, each eigenvalue of $L'$ is either near 0 or near 1 and hence $P(L')$ is a projection with the same rank as $L$.

Thus, to smoothen $E$, let $L_z$ be the orthogonal projection onto $E_z$ for $z \in M$; we approximate $L$ by a $C^k$-map $L': M \to \text{Lin}_C(C^n, C^n)$ and let $E'$ be the bundle (2.1) associated to $P(L')$. By (2.2) the difference equals

$$P(L') - L = \frac{1}{2\pi i} \int_C ((\zeta I - L')^{-1} - (\zeta I - L)^{-1}) \, d\zeta$$

and is $C^\ell$-small when $L' - L$ is.

In the real case we extend $L: \mathbb{R}^n \to \mathbb{R}^n$ to a complex linear map $L: C^n \to C^n$ and observe that $P(L)$ is also real (i.e., it maps $\mathbb{R}^n$ to itself) when $C$ is the curve (2.3). Hence the restriction of $P(L)$ to $\mathbb{R}^n$ solves the problem.
Let now $M$ be a totally real submanifold of $\mathbb{C}^n$ and $E \to M$ a $C^r$ rank $r$ complex subbundle of $M \times \mathbb{C}^n$. For each $z \in M$ let $L_z: \mathbb{C}^n \to E_z$ be the orthogonal projection onto $E_z$. We may approximate the $C^r$-map $L: M \to \text{Lin}_C(\mathbb{C}^n, \mathbb{C}^n)$ as well as we like in the $C^r$-topology on $M$ by the restriction to $M$ of a holomorphic map $L': U \to \text{Lin}_C(\mathbb{C}^n, \mathbb{C}^n)$ defined on an open neighborhood $U \subset \mathbb{C}^n$ of $M$. Shrinking $U$ we may assume that $L'_z$ has exactly $r$ eigenvalues inside $C$ (2.3) for each $z \in U$, so $P(L'_z)$ is a rank $r$ projection. The map $z \to P(L'_z)$ is holomorphic in $U$ and determines a holomorphic rank $r$ vector bundle $E'$ over $U$, with $E'|_M$ close to $M$. ♣

Let $d = \partial + \overline{\partial}$ be the splitting of the exterior derivative on a complex manifold.

**Definition 3.** ($\overline{\partial}$-flat functions) If $M$ is a closed subset in a complex manifold $X$ and $u$ is a $C^k$-function ($k \geq 1$) defined in a neighborhood of $M$ in $X$, we say that $u$ is $\overline{\partial}$-flat (to order $k$) on $M$ if $\partial^a(\overline{\partial}u)(z) = 0$ for each $z \in M$ and each derivative $\partial^a$ of total order $|\alpha| \leq k - 1$ with respect to the underlying real local coordinates on $X$.

We shall commonly use the phrase ‘$u$ is a $\overline{\partial}$-flat $C^k$-function’ when it is clear from the context which subset $M \subset X$ is meant.

**2.3 Lemma.** ($\overline{\partial}$-flat partitions of unity) Let $M$ be a totally real $C^k$-submanifold of a complex manifold $X$; $k \geq 1$. For every open covering $\mathcal{U}$ of $M$ in $X$ there exists a $C^k$ partition of unity on a neighborhood of $M$ in $X$, subordinate to the covering $\mathcal{U}$ and consisting of functions that are $\overline{\partial}$-flat to order $k$ on $M$.

**Proof.** We may assume that $\mathcal{U}$ consists of coordinate neighborhoods. Let $\phi^0_\nu$ be a $C^k$ partition of unity subordinate to $\mathcal{U}|_M = \{ U \cap M: U \in \mathcal{U} \}$. We may assume that the index sets agree, so supp $\phi^0_\nu \subset U_\nu$ for each $\nu$. By passing to local coordinates we may find a $\overline{\partial}$-flat $C^k$ extension $\phi_\nu$ of $\phi^0_\nu$ with supp $\phi_\nu \subset U_\nu$. Since $\rho = \sum_\nu \phi_\nu = 1$ on $M$, $\rho \neq 0$ in a neighborhood $V$ of $M$ in $X$. It is immediate that $\phi_\nu = \phi^0_\nu/\rho$ is a $C^k$ partition of unity on $V$ which is $\overline{\partial}$-flat (to order $k$) on $M$.

As a consequence of lemma 2.3 we see that the usual results about $\overline{\partial}$-flat extensions of maps into $\mathbb{C}^N$ are also valid for totally real submanifolds in arbitrary complex manifolds.

**2.4 Lemma.** (Asymptotic complexifications) Let $M$ be a totally real $C^k$-submanifold of $\mathbb{C}^n$ of real dimension $m \leq n$; $k \geq 1$. Then there exists a $C^k$-submanifold $\tilde{M} \supset M$ in $\mathbb{C}^n$, of real dimension $2m$, with the following property: $\tilde{M}$ may be covered by $C^k$ local parametrizations $Z: U \to Z(U) \subset \tilde{M}$, with $U \subset \mathbb{C}^m$ open subsets, such that $Z^{-1}(M) = U \cap \mathbb{R}^m$ and $Z$ is $\overline{\partial}$-flat on $U \cap \mathbb{R}^m$. Moreover there is a $C^k$-retraction of a neighborhood of $M$ in $\mathbb{C}^n$ onto $\tilde{M}$ which is $\overline{\partial}$-flat on $M$.

**Proof.** By a theorem of Whitney ([Wh2], Theorem 1, p. 654) there exists a $C^\omega$-manifold $M_0$ and a $C^k$-diffeomorphism $G^0: M_0 \to M$. The manifold $M_0$ has a complexification $\tilde{M}_0$ which is a complex manifold containing $M_0$ as a maximal real submanifold. The map $G^0$ has a $\overline{\partial}$-flat extension $G: \tilde{M}_0 \to \mathbb{C}^n$ which is an injective immersion at $M$. (To obtain $G$ it suffices to patch local $\overline{\partial}$-flat extensions of $G^0$ by a $\overline{\partial}$-flat partition of unity provided by lemma 2.3). Hence $G$ maps a neighborhood of $M_0$ in $\tilde{M}_0$ diffeomorphically onto its image.
$\widetilde{M} \subset \mathbb{C}^n$. When $Z^0: U \to \widetilde{M}_0$ ($U$ open in $\mathbb{C}^m$) is a local holomorphic parametrization with $(Z^0)^{-1}(M_0) = U \cap \mathbb{R}^m$, the map $Z = G \circ Z^0: U \to \widetilde{M}$ is a local parametrization of the type described in lemma 2.4. Note that $T_z\widetilde{M} = T_z^c M$ for each $z \in M$.

Next we prove the existence of a retraction onto $\widetilde{M}$ which is $\overline{\partial}$-flat on $M$. Let $\nu \to M$ be the complex normal bundle of $M$ in $\mathbb{C}^n$. By lemma 2.2 there is an open neighborhood $O$ of $M$ in $\mathbb{C}^n$ and a holomorphic rank $(n - m)$ subbundle $N \subset O \times \mathbb{C}^n$ such that $N|_M$ approximates $\nu$ well. Shrinking $O$ we may assume that $N$ is transversal to $\widetilde{M}$ in $O$. This means that the map $\phi: N|_{\widetilde{M}} \to \mathbb{C}^n$, $\phi(z, v) = z + v$, is a $C^k$-diffeomorphism from a neighborhood $W$ of the zero section in $N|_{\widetilde{M}}$ onto its image $O_0 \subset O \subset \mathbb{C}^n$. We may assume that $W \cap N_z$ is starshaped with respect to $0_z \in N_z$ for each $z \in \widetilde{M}$. Now the deformation retraction $(z, v) \to (z, tv)$ $(t \in [0, 1])$ of $W$ onto the zero section in $N|_{\widetilde{M}}$ may be transported by $\phi$ to a retraction $F: [0, 1] \times O_0 \to O_0$ of $O_0$ onto the submanifold $\widetilde{M} \cap O_0$. Set $\pi = F_0: O_0 \to \widetilde{M} \cap O_0$. Let $U \subset \mathbb{C}^m$ and let $Z: U \to \widetilde{M}$ be a local $C^k$-parametrization such that $Z(U \cap \mathbb{R}^m) \subset M$ and $Z$ is $\overline{\partial}$-flat on $U \cap \mathbb{R}^m$. Choose holomorphic sections $s_1, \ldots, s_{n-m}$ of $N$ which provide a trivialization of $N$ near $Z(U)$. Then

$$(z', w') \to Z(z') + \sum_{j=1}^{n-m} w'_j s_j(Z(z'))$$

is a $C^k$-diffeomorphism of a neighborhood $W$ of $U \times \{0\}^{n-m}$ in $\mathbb{C}^n$ onto $\pi^{-1}(Z(U))$, and it is $\overline{\partial}$-flat on $(U \cap \mathbb{R}^m) \times \{0\}^{n-m}$. In these coordinates the maps $F_t$ are given by $(z', w') \to (z', tw')$, hence $F_t$ is $\overline{\partial}$-flat on $\pi^{-1}(M)$. ♠

2.5 Lemma. (The rough multiplication) Let $U$ be an open set in $\mathbb{R}^N$, $f \in C^k(U)$ and $g \in C^{k-1}(U)$, where $k \geq 1$. Let $E$ be a closed subset of $U$ such that $f(x) = 0$ for all $x \in E$. Then there exists a function $h \in C^k(U)$ such that

(i) $|\partial^\alpha(h - f g)| = o(d_E^{-|\alpha|})$ for $|\alpha| < k$, uniformly on compacts in $U$,

(ii) at points of $E$ we have $\partial^\alpha h = \sum_{0 \neq \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha - \beta} g$ for $|\alpha| \leq k$, and

(iii) if $U \subset \mathbb{C}^N$ and if $f$ and $g$ as above are $\overline{\partial}$-flat on $E \subset f^{-1}(0)$, then so is $h$.

The proof is similar to the better known ‘Glaeser-Kneser rough composition theorem’; the main point is to verify that the collection of functions $(\partial^\alpha h)|_{|\alpha| \leq k}$ on $E$, defined by (ii), are a Whitney system, i.e., they satisfy the assumptions of the Whitney’s extension theorem (see [Wh1] or [T]). We shall leave out the details of this verification. Let $h$ be a $C^k$-function provided by Whitney’s theorem, having the partial derivatives given by (ii) at points of $E$. Then (i) follows easily by comparing the Taylor expansions of $\partial^\alpha h$, $\partial^\beta f$ and $\partial^{\alpha - \beta} g$ about the nearest point in $E$. The case (iii) follows from (ii) which is seen as follows. From (ii) we get at points of $E$ and for $|\alpha| \leq k - 1$

$$\partial^\alpha h = \partial^\alpha(f \cdot g) + \sum_{0 \neq \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha - \beta} g.$$ 

If $f$ and $g$ are $\overline{\partial}$-flat on $E$, this expression vanishes when $|\alpha| \leq k - 1$, so we get (iii). ♠

The following lemma is needed in the proof of theorem 1.4 and its corollaries.

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2.6 Lemma. Let $M$ be a totally real, $m$-dimensional $C^k$-submanifold of $\mathbb{C}^n$, $f: M \to \mathbb{C}^p$ a $C^k$-map and $l: M \to \text{Lin}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^p)$ a $C^{k-1}$-map such that for each $z \in M$, $l_z$ agrees with $df_z$ on $T_zM$. Then there is a neighborhood $U \subset \mathbb{C}^n$ of $M$ and a $C^k$-map $F: U \to \mathbb{C}^p$ which is $\overline{\partial}$-flat on $M$ and satisfies $F(z) = f(z)$ and $dF_z = l_z$ for all $z \in M$.

Proof. It suffices to prove the result for functions $(p = 1)$; the general case then follows by applying it componentwise. So we shall assume $p = 1$.

We first consider the local case. Fix a point $z_0 \in M$. Choose $e_1, \ldots, e_{n-m} \in \mathbb{C}^n$ such that these vectors, together with the tangent space $T_{z_0}M$, span a totally real subspace of $T_{z_0} \mathbb{C}^n$ of maximal dimension $n$. When $\kappa: U \to M$ is a $C^k$-parametrization of a small neighborhood of $z_0$ in $M$, with $\kappa(0) = z_0$, and $V$ is a sufficiently small neighborhood of 0 in $\mathbb{R}^{n-m}$, the map $\phi(x, y) = \kappa(x) + \sum_{j=1}^{n-m} y_j e_j$ ($x \in U, \ y \in V$) is a $C^k$-diffeomorphism onto an $n$-dimensional totally real submanifold in $\mathbb{C}^n$. Observe that for $x \in U$ and $(u, v) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ we have

$$l_{\kappa(x)} \circ d\phi(x, 0)(u, v) = df_{\kappa(x)} \circ d\kappa_x(u) + \sum_{j=1}^{n-m} v_j l_{\kappa(x)}(e_j).$$

Since $l_{\kappa(x)}(e_j)$ is only of class $C^{k-1}$ in $x$, we apply the rough multiplication lemma to the pairs $y_j l_{\kappa(x)}(e_j)$ for $1 \leq j \leq n-m$ to get a $C^k$ function $h$ on $U \times V$ satisfying $\frac{\partial h}{\partial y_j}(x, 0) = 0$, $\frac{\partial h}{\partial y_j}(x, 0) = l_{\kappa(x)}(e_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n-m$. With $F^0(x, y) = f(\kappa(x)) + h(x, y)$ it follows that $dF^0_{(x, 0)} = l_{\kappa(x)} \circ d\phi(x, 0)$. When $F^0$ resp. $\phi$ are $C^k$-extensions of $F^0$ resp. $\phi$ which are $\overline{\partial}$-flat on $\mathbb{R}^m$, we see that $\phi$ is a $C^k$-diffeomorphism of a neighborhood of 0 in $\mathbb{C}^n$ onto a neighborhood of $z_0 \in \mathbb{C}^n$. Thus, near $z_0$, $F = F^0 \circ \phi^{-1}$ is a $C^k$ $\overline{\partial}$-flat extension of $f$. When $z \in M$ we have $dF_z = l_z$ on a maximal totally real subspace, so these two linear maps are equal on $T_z \mathbb{C}^n$. This establishes the local case.

For the global case let $\mathcal{U} = \{U_i\}$ be an open covering of $M$ and $F^{(i)}$ a $\overline{\partial}$-flat extension of $f$ in $U_i$, with $dF^{(i)}_z = l_z$ for $z \in U_i \cap M$. By lemma 2.3 there is a partition of unity $\{\phi_i\}$ by $\overline{\partial}$-flat $C^k$-functions on a neighborhood of $M$ subordinate to $\mathcal{U}$. We set $F = \sum_i \phi_i F^{(i)}$, where the term with index $i$ is zero outside $U_i$. When $z \in M$, $dF_z = \sum_i \phi_i d\phi_i(z) dF^{(i)}_z + \sum_i f(z) d\phi_i(z)$. Since $\sum_i \phi_i = 1$, $\sum_i d\phi_i = 0$ and we get $dF_z = l_z$. ♦

§3. Solving the $\overline{\partial}$-equation in tubes around totally real manifolds.

In this section we construct a family of integral kernel, depending on a parameter $\delta > 0$, for solving the $\overline{\partial}$-equation in tubes $T_{\delta}M$ around compact totally real submanifolds $M \subset \mathbb{C}^n$ of class $C^1$. The main result is theorem 3.1 which is identical with theorem 1.1 except that it contains additional Hölder estimates (3.3) and (3.4).

We denote by $d_M$ the Euclidean distance to $M$. If $M$ is of class $C^k$, it is well known that $\rho = d_M^2$ is a $C^k$ strictly plurisubharmonic function in a neighborhood of $M$ when $k > 1$, and when $k = 1$ there is a strictly plurisubharmonic $C^2$-function $\rho$ such that $\rho = d_M^2 + o(d_M^2)$. As in sect. 1 let $\mathcal{T}_\delta$ denote the tubular neighborhood of $M$ of radius $\delta$, i.e., the set of points whose distance to $M$ is less than $\delta$. 12
For a domain $D$ in $\mathbb{R}^n$ (or in $\mathbb{C}^n$), a bounded function $u$ in $D$ belongs to the Hölder class $\Lambda^s(D)$ for some $0 < s < 1$ if $|u|_{s,D} = \sup\{|u(z+h) - u(z)|/h|^{s}; h \neq 0, z, z+h \in D\} < \infty$; in this case the Hölder $s$-norm of $u$ is defined by $\|u\|_{\Lambda^s(D)} = \|u\|_{L^\infty(D)} + |u|_{s,D}$. When $s = 1$ we set $|u|_{1,D} = \sup\{|u(z+h) + u(z-h) - 2u(z)|/|h|^{1}; h \neq 0, z, z-h, z+h \in D\}$; $\Lambda^1(D)$ is called the Zygmund class on $D$. If $D$ is a tubular neighborhood $T_\delta M$ of a submanifold $M$, we write $|u|_{s,\delta}$ for $|u|_{s,T_\delta M}$. When $s = k + \alpha$, $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, we take $\|u\|_{\Lambda^\alpha(D)} = \|u\|_{\Lambda^{k+\alpha}(D)} + |D^k u|_{\alpha,D}$. We sometimes write $C^{k+\alpha}(D)$ for $\Lambda^{k+\alpha}(D)$ when $0 < \alpha < 1$.

We extend function space norms to vector fields or differential forms on open sets in $\mathbb{R}^n$ as the sum of the norms of the components. When $M$ is a compact $C^k$-manifold, we define the norms on functions or forms on $M$ as follows: Let $\Phi_j: U_j \to V_j \subset M$, $j = 1, \ldots, p$, be a covering of $M$ by local parametrizations, and $\{\phi_1, \ldots, \phi_p\}$ a $C^k$ partition of unity subordinate to the covering $\{V_1, \ldots, V_p\}$ of $M$. Then we set $\|u\| = \sum_{j=1}^p \|\Phi_j^* (\phi_j u)\|$, where $\|\cdot\|$ is a Hölder or some other function space norm. Different choices of $\{\Phi_j\}$ and $\{\phi_j\}$ give rise to equivalent norms on the same space.

Let $z = (z_1, \ldots, z_n)$ be the complex coordinates and $(x_1, y_1, \ldots, x_n, y_n)$ $(z_j = x_j + iy_j)$ the underlying real coordinates on $\mathbb{C}^n = \mathbb{R}^{2n}$. For $1 \leq j \leq 2n$, $\partial_j$ denotes the partial derivative with respect to the $j$-th variable. If $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multiindex of length $2n$ then $\partial^\alpha$ denotes the corresponding partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$ with respect to the real variables on $\mathbb{C}^n = \mathbb{R}^{2n}$.

If $f$ is a function or a form near $M$, we shall say that $f$ vanishes to order $l$ on $M$ if $|f(z)| = o(d_M(z)^l)$ and $\partial^\alpha f = 0$ on $M$ when $|\alpha| \leq l$. Recall that any $C^k$-function $f$ on $M$ can be extended to a $C^k$-function on $\mathbb{C}^n$ such that $\overline{\partial} f$ vanishes to order $k - 1$ on $M$ (Lemma 4.3 in [HöW]).

We call a continuous function $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ a modulus of continuity if it is non-decreasing, sub-additive, and $\omega(0) = 0$. If $f: A \to C$, $A \subset \mathbb{R}^n$, is uniformly continuous, we define the modulus of continuity of $f$ by $\omega(f,t) = \sup\{|f(x) - f(y)|; |x - y| \leq t\}$, $t \geq 0$. $\omega(f, \cdot)$ is clearly a modulus of continuity as defined above. We say that a modulus of continuity $\omega$ is a modulus of continuity for a function $f$ if $\omega(f,t) \leq \omega(t)$ for all $t \geq 0$. If $f$ is a form on $A$, $\omega(f, t)$ is defined as the sum of the moduli of continuity of its components.

We denote by $C^l_{p,q}(U)$ the space of $(p,q)$-forms of class $C^l$ on an open set $U \subset \mathbb{C}^n$.

### 3.1 Theorem
Let $M \subset \mathbb{C}^n$ be a closed totally real $C^1$-submanifold and let $0 < c < 1$. Denote by $T_\delta$ the tube of radius $\delta > 0$ around $M$. Then there exist constants $\delta_0 > 0$ and $C > 0$ such that the following holds for $0 < \delta \leq \delta_0$, $l \geq 1$, $p \geq 0$, $q \geq 1$: For each $u \in C^l_{p,q}(T_\delta)$ with $\overline{\partial} u = 0$ there is a $v \in C^l_{p,q-1}(T_\delta)$ satisfying $\overline{\partial} v = u$ in $T_{c\delta}$ and

$$||\partial^\alpha v||_{L^\infty(T_{c\delta})} \leq C (\delta ||\partial^\alpha u||_{L^\infty(T_\delta)} + \delta^{1-|\alpha|} ||u||_{L^\infty(T_\delta)})$$

(3.1)

In particular we have $||v||_{L^\infty(T_{c\delta})} \leq C\delta ||u||_{L^\infty(T_\delta)}$. If $q = 1$ and the equation $\overline{\partial} v = u$ has a solution $v_0 \in C^{l+1}_{p,0}(T_\delta)$, there is a solution $v \in C^{l+1}_{p,0}(T_\delta)$ of $\overline{\partial} v = u$ satisfying for $1 \leq j \leq n$

$$||\partial_j \partial^\alpha v||_{L^\infty(T_{c\delta})} \leq C (\omega(\partial_j \partial^\alpha v_0, \delta) + \delta^{1-|\alpha|} ||u||_{L^\infty(T_\delta)})$$

(3.2)
If we assume in addition that $\partial^\alpha u \in \Lambda^s(\mathcal{T}_\delta)$ for some $|\alpha| \leq l$ and $0 < s \leq 1$, we may choose $v$ as above satisfying also the following estimates (with a constant $C_u$ depending only on $s$ and $c$):

\[
\|\partial_j \partial^\alpha v\|_{L^\infty(\mathcal{T}_\delta)} \leq C_s \left( \delta^s \|\partial^\alpha u\|_{\Lambda^s(\mathcal{T}_\delta)} + \delta^{-|\alpha|} \|u\|_{L^\infty(\mathcal{T}_\delta)} \right) \quad (3.3)
\]

\[
\|\partial_j \partial^\alpha v\|_{\Lambda^s(\mathcal{T}_\delta)} \leq C_s \left( \|\partial^\alpha u\|_{\Lambda^s(\mathcal{T}_\delta)} + \delta^{-|\alpha|-s} \|u\|_{L^\infty(\mathcal{T}_\delta)} \right) \quad (3.4)
\]

Remarks. 1. If $u$ is of class $C^l$, there is in general no $C^{l+1}$ solution $v$ to $\bar{\partial} v = u$.

2. In (3.3) one may be tempted to delete $\delta^s$ and use instead the $L^\infty(\mathcal{T}_\delta)$ norm of $\partial^\alpha u$ in the first term on the right hand side. This however is false even when $u = 1$ and is a well known phenomenon. Since the Bochner-Martelli operator used in the proof is a homogeneous convolution operator, it gains one derivative in norms such as Hölder, Zygmund, Sobolev, but not in the sup-norm or the $C^l$-norm.

3. Theorem 3.1 has the following extension to non-closed totally real $C^1$-submanifolds $M'$ in $\mathbb{C}^n$. Let $K$ be a compact subset of $M'$ and let $K' \subset M'$ be a compact neighborhood of $K$ in $M'$. For $\delta > 0$ we set

\[ U_\delta = \{ z \in \mathbb{C}^n : d_K(z) < \delta \}, \quad U'_\delta = \{ z \in \mathbb{C}^n : d_{K'}(z) < \delta \}. \]

Choose $c \in (0, 1)$. Given a form $u \in C^l_{p,q}(U')$ with $\bar{\partial} u = 0$, we can solve $\bar{\partial} v = u$ in $U_{c\delta}$, and the estimates in theorem 3.1 remain valid when the tube $T_{c\delta}$ is replaced by $U_{c\delta}$ on the left hand side, and $T_\delta$ is replaced by $U'_\delta$ on the right hand side of each estimate. The proof can be obtained by simple modifications of the kernel construction below. This applies to compact totally real submanifolds with boundary in $\mathbb{C}^n$ since any such is a compact domain in a larger totally real submanifold.

In this section $C$ denotes some constant whose value may change every time it occurs, but which does not depend on quantities such as $u$, $\delta$ etc.

For $(0,1)$-forms $u$ (and hence for $(p,1)$-forms, $0 \leq p \leq n$) a large part of the result comes from the interior elliptic regularity of the $\bar{\partial}$-operator and has nothing to do with the particular solution $v$:

3.2 Lemma. Let $0 < c < 1$. There exists a constant $C > 0$ satisfying the following. If $K \subset \mathbb{C}^n$ is a compact subset, $\mathcal{T}_\delta = \{ z : d(z,K) < \delta \}$, and $u$ a continuous function in $\mathcal{T}_\delta$ such that $\bar{\partial} v \in C^l_{(0,1)}(\mathcal{T}_\delta)$, then $v \in C^l(\mathcal{T}_\delta)$ and

\[
\|\partial^\alpha v\|_{L^\infty(\mathcal{T}_\delta)} \leq C(\delta \|\partial^\alpha \bar{\partial} v\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{-|\alpha|} \|v\|_{L^\infty(\mathcal{T}_\delta)}); \quad |\alpha| \leq l.
\]

If $\bar{\partial} v = \bar{\partial} f$ for some $f \in C^{l+1}(\mathcal{T}_\delta)$, then $v$ is also $C^{l+1}$ and satisfies

\[
\|\partial_j \partial^\alpha v\|_{L^\infty(\mathcal{T}_\delta)} \leq C(\omega(\partial_j \partial^\alpha f, \delta) + \delta^{-l-1} \|v\|_{L^\infty(\mathcal{T}_\delta)}); \quad |\alpha| = l.
\]
Proof. We apply the Bochner-Martinelli formula

$$g(z) = \int_{\partial T_\delta} g(\zeta) B(\zeta, z) - \int_{T_\delta} \overline{\theta g(\zeta)} \wedge B(\zeta, z),$$

valid for $g \in C^1(\overline{T}_\delta)$, where $B(\zeta, z)$ is the Bochner-Martinelli kernel

$$B(\zeta, z) = c_n \sum_{j=1}^{n} (-1)^{j-1} \overline{\frac{\zeta_j - z_j}{||\zeta - z||^{2n}}} d\zeta[j] \wedge d\zeta$$

which is a closed integrable $(n, n - 1)$-form. Let $z \in T_\delta$ and let $\chi: \mathbb{R} \to [0, 1]$ be a cutoff function with $\chi(t) = 1$ when $|t| \leq \frac{1}{2}$ and $\chi(t) = 0$ when $|t| \geq 1$. For $w \in \mathbb{C}^n$ set $\chi_\delta(w) = \chi(\frac{|w|}{|1 - \frac{1}{2}|\delta})$. It follows that the partial derivatives satisfy $|\partial^\alpha \chi_\delta| \leq C_\alpha \delta^{-|\alpha|}$ for all $\alpha$. Applying the Bochner-Martinelli formula to $g(\zeta) = \chi_\delta(\zeta - z)v(\zeta)$ we obtain

$$v(z) = -\int_{T_\delta} \overline{\theta \chi_\delta(\zeta - z)v(\zeta)} \wedge B(\zeta, z)$$

$$= -\int_{T_\delta} \overline{\theta v(\zeta)} \wedge \chi_\delta(\zeta - z)B(\zeta, z) - \int_{T_\delta} v(\zeta) \overline{\theta \chi_\delta(\zeta - z)} \wedge B(\zeta, z)$$

$$= I_1(z) + I_2(z). \quad (3.5)$$

These are convolution operators and we may differentiate on either integrand. This gives for $|\alpha| \leq l$:

$$\partial^\alpha v(z) = \partial^\alpha I_1(z) + \partial^\alpha I_2(z)$$

$$= -\int_{T_\delta} \partial^\alpha \overline{\theta v(\zeta)} \wedge \chi_\delta(\zeta - z)B(\zeta, z) - \int_{T_\delta} v(\zeta) \partial^2 \partial^\alpha \overline{\theta \chi_\delta(\zeta - z)} \wedge B(\zeta, z)).$$

Setting $c' = \frac{1 - c}{2}$, $c'' = \frac{1}{2} c'$ and using $|B(\zeta, z)| \leq C |\zeta - z|^{1-2n}$ we can estimate the integrals for $|\alpha| \leq l$ as follows:

$$|\partial^\alpha I_1(z)| \leq C \int_{|\zeta - z| \leq c' \delta} |\partial^\alpha \overline{\theta v(\zeta)}| |\zeta - z|^{1-2n} dV$$

$$\leq C ||\partial^\alpha \overline{\theta v}||_{L^{\infty}(T_\delta)} \int_{|\zeta - z| \leq c' \delta} |\zeta - z|^{1-2n} dV$$

$$\leq C ||\partial^\alpha \overline{\theta v}||_{L^{\infty}(T_\delta)} \int_{0}^{c' \delta} \frac{1-2n-1}{r^{2n-1}} \leq C \delta ||\partial^\alpha \overline{\theta v}||_{L^{\infty}(T_\delta)},$$

$$|\partial^\alpha I_2(z)| \leq C \int_{c' \delta \leq |\zeta - z| \leq c' \delta} |v(\zeta)| \delta^{-2n-|\alpha|} dV \leq C ||v||_{L^{\infty}(T_\delta)} \delta^{-|\alpha|}.$$
This proves the first estimate in lemma 3.2. The estimate for $|\partial^\alpha I_2|$ also holds for derivatives of order $|\alpha| = l + 1$.

We now assume that $\overline{\partial} v = \overline{f}$ for some $f \in C^{l+1}(\Omega_\delta)$; then $v - f$ is holomorphic and hence $v$ is also $C^{l+1}$. We wish to estimate the derivatives of order $l + 1$ of $I_1(z)$. For $|\alpha| = l$ we have

$$\partial_j \partial^\alpha I_1(z) = -\partial_j \int_{T_\delta} \partial^\alpha \overline{\partial} v(\zeta) \wedge \chi_\delta(\zeta - z) B(\zeta, z).$$

We now apply (3.5) to $f$, replacing $\overline{\partial} f$ by $\overline{\partial} v = \overline{f}$ in the first term on the right hand side and differentiating under the integral, to get

$$\partial_j \partial^\alpha f(z) = -\partial_j \int_{T_\delta} \partial^\alpha \overline{\partial} v(\zeta) \wedge \chi_\delta(\zeta - z) B(\zeta, z)$$

$$- \int_{T_\delta} \partial_j \partial^\alpha f(\zeta) \wedge \overline{\partial} \chi_\delta(\zeta - z) \wedge B(\zeta, z).$$

Observe that the first term on the right hand side equals $\partial_j \partial^\alpha I_1(z)$ from the previous display. For a fixed $z \in \mathbb{C}^n$ we also apply (3.5) to the constant function $\partial_j \partial^\alpha f(z)$:

$$\partial_j \partial^\alpha f(z) = - \int_{T_\delta} \partial_j \partial^\alpha f(z) \overline{\partial} \chi_\delta(\zeta - z) \wedge B(\zeta, z).$$

Combining the above three formulas we get

$$\partial_j \partial^\alpha I_1(z) = \int_{T_\delta} (\partial_j \partial^\alpha f(\zeta) - \partial_j \partial^\alpha f(z)) \overline{\partial} \chi_\delta(\zeta - z) \wedge B(\zeta, z)$$

and hence $|\partial_j \partial^\alpha I_1(z)| \leq C \omega(\partial_j \partial^\alpha f, \delta)$.

For lemma 3.2 it follows that the estimates (3.1) and (3.2) in theorem 3.1 will be proved for $(p,1)$-forms $u$ if we can find a solution $v$ which satisfies a sup-norm estimate $\|v\|_{L^\infty(\Omega_\delta)} \leq C \delta \|u\|_{L^\infty(\Omega_\delta)}$. Such a solution is obtained by a linear operator given by an integral kernel that we now construct.

**Construction of the kernel for $(0,1)$-forms.** We shall use the Koppelman’s formula which we now recall. For $v, w \in \mathbb{C}^n$ let $<v, w> = \sum_{i=1}^n v_i w_i$. Let $V \subset \mathbb{C}^{2n}$ and $\Omega' \subset \Omega \subset \mathbb{C}^n$ be open subsets such that $\Omega$ has piecewise $C^1$-boundary and $\overline{\Omega} \times \Omega \subset V$. Let $P = P(\zeta, z) = (P_1, \ldots, P_n) : V \rightarrow \mathbb{C}^n$ be a $C^1$-map satisfying

(i) $P(\zeta, z) = \overline{\zeta - z}$ in a neighborhood of the diagonal of $\Omega' \times \Omega'$, and

(ii) the function $\Phi : V \rightarrow \mathbb{C}$, $\Phi(\zeta, z) = <P(\zeta, z), \zeta - z>$, satisfies $\Phi(\zeta, z) \neq 0$ when $z \in \Omega'$ and $\zeta \in \overline{\Omega} \setminus \{z\}$.

Any such map $P$ is called a *Leray map* for the pair $\Omega' \subset \Omega$ and $\Phi$ is the corresponding *support function*. We shall use the notation

$$d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

$$\overline{\partial}_\zeta P[i] = \overline{\partial}_\zeta P_1 \wedge \cdots \overline{\partial}_\zeta P_i \cdots \wedge \overline{\partial}_\zeta P_n,$$

$$\overline{\partial}_\zeta P[j] = \overline{\partial}_\zeta P_1 \wedge \cdots \overline{\partial}_\zeta P_j \cdots \wedge \overline{\partial}_\zeta P_n.$$
Define the integral kernels

\[ K(\zeta, z) = c_n \Phi(\zeta, z)^{-n} \sum_{j=1}^{n} (-1)^{j-1} P_j \partial_{\zeta} [j] \wedge d\zeta, \]
\[ L(\zeta, z) = c_n \Phi(\zeta, z)^{-n} \sum_{i \neq j} (-1)^{i+j} P_j \partial_{\zeta} [i] \wedge \partial_{\zeta} [j] \wedge d\zeta. \]

Note that the kernel \( K(\zeta, z) \) is locally integrable when \( z \in \Omega' \). It is also important to observe that, if \( a(\zeta, z) \) is a \( C^1 \) function, the kernels generated by \( P \) resp. \( aP \) are identical outside the zero set \( a = 0 \). For a suitable choice of the constant \( c_n \in \mathbb{R} \) we then have the following Koppelman-Leray representation formula for \( \partial_{\zeta} \)-closed \((0, 1)\)-forms \( u \in C^1_{0, 1}(\overline{\Omega}) \):

\[ u(z) = \int_{\partial_{\Omega}} L(\zeta, z) \wedge u(\zeta) + \partial_{\zeta} \int_{\Omega} K(\zeta, z) \wedge u(\zeta), \quad z \in \Omega'. \quad (3.6) \]

This follows by applying the Stokes formula to the first integral on the right hand side to transfer the integration to an \( \epsilon \)-sphere around \( z \) and using \( \partial_{\zeta} K = -\partial_{\zeta} L \); in the limit as \( \epsilon \to 0 \) we obtain (3.6) by a usual residue calculation. For \( \zeta \) near \( z \), the kernel \( L \) coincides with the B-M kernel for \((0, 1)\)-forms; in fact, for the Leray map \( P(\zeta, z) = \zeta - z \), \((3.6) \) is the classical Bochner-Martinelli-Koppelman formula.

We have a lot of freedom in the choice of the map \( P \) which determines \( K \) and \( L \). If we choose it such that \( P(\zeta, \cdot) \) is holomorphic in \( \Omega' \) when \( \zeta \in \partial \Omega \), then \( L(\zeta, z) = 0 \) for such \( \zeta \) and \( z \) (since each term in \( L \) contains a derivative \( \partial_{\zeta} [i] \)), and hence the function

\[ v(z) = \int_{\Omega} K(\zeta, z) \wedge u(\zeta) \quad (3.7) \]

solves the equation \( \partial_{\zeta} v = u \) in \( \Omega' \).

We shall construct the integral kernel of our solution operator on \( \mathcal{T}_0 \) by combining the Bochner-Martinelli kernel \( \zeta - z \) near the diagonal \( \zeta = z \) of the smaller tube \( \mathcal{T}_{0, \delta} \) with the Henkin kernel when \( \zeta \) is near the boundary of \( \mathcal{T}_0 \) and \( z \in \mathcal{T}_{0, \delta} \). This will give a family of linear solution operators of the form (3.7) depending on \( \delta \) for small \( \delta > 0 \).

Let \( \rho \) be the strongly plurisubharmonic function mentioned in the beginning of this section. Since \( \{ \rho < (1 - \epsilon) \delta^2 \} \subset \mathcal{T}_0 \subset \{ \rho < (1 + \epsilon) \delta^2 \} \) for sufficiently small \( \delta > 0 \), we may replace the tube \( \mathcal{T}_0 \) with the sublevel sets \( \{ \rho < \delta^2 \} \) which we still denote by \( \mathcal{T}_0 \).

The construction of the kernel will proceed through several lemmas. First we recall from [HL] the following well known result about the existence of the Henkin support function \( \Phi \) and the corresponding Leray map \( P \) on a fixed strongly plurisubharmonic domain which in our case is a tube \( \mathcal{T}_{0, \delta_0} \) of some fixed (small) radius \( \delta_0 > 0 \).

3.3 Lemma. There exist constants \( C, R > 0 \) such that, for \( \delta_0 > 0 \) sufficiently small, there are functions \( \Phi(\zeta, z) \) and \( A(\zeta, z) \) in \( C^1(\mathcal{T}_{0, \delta} \times \mathcal{T}_{0, \delta}) \), with \( \Phi \) holomorphic in \( z \), and there is a \( C^1 \)-function \( B(\zeta, z) \), defined for \( \zeta, z \in \mathcal{T}_{0, \delta} \) and \( |\zeta - z| \leq R \), satisfying the following:
(i) $\Phi(\zeta, z) = A(\zeta, z)B(\zeta, z)$,
(ii) $|B(\zeta, z)| \geq C$ and $\text{Re}A(\zeta, z) \geq \rho(\zeta) - \rho(z) + C|\zeta - z|^2$ when $|\zeta - z| \leq R$,
(iii) $|\Phi(\zeta, z)| \geq C$ when $|\zeta - z| \geq \frac{R}{2}$, and
(iv) with $\Phi$ as above, there exists a map $P = P(\zeta, z) = (P_1, \cdots, P_n)$ such that for all $j$, $P_j \in C^1(\mathcal{T}_{\delta_0} \times \mathcal{T}_{\delta_0})$, $P_j$ is holomorphic in $z$, and $\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z \rangle$.

Proof. This follows from the proof of Theorems 2.4.3 and 2.5.5. in [HL]. $A(\zeta, z)$ is an approximate Levi polynomial in $z \in \mathbb{C}^n$ of the form

$$A(\zeta, z) = 2 \sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{j,k=1}^{n} a_{jk}(\zeta) (\zeta_j - z_j)(\zeta_k - z_k),$$

where $a_{jk}$ are $C^1$ functions which approximate the partial derivatives $\partial^2 \rho/\partial \zeta_j \partial \zeta_k$ sufficiently well on $\mathcal{T}_{\delta_0}$ (Lemma 2.4.2 in [HL]). In fact, when $\rho$ is of class $C^3$ or better, we might simply take $a_{jk} = \partial^2 \rho/\partial \zeta_j \partial \zeta_k$.

The only small change from [HL] is that, in our situation, the maps $\Phi$ and $P$ may be defined globally for $\zeta \in \mathcal{T}_{\delta_0}$, and not only for $\zeta$ near the boundary of $\mathcal{T}_{\delta_0}$, provided that $\delta_0 > 0$ is sufficiently small. This follows from the thinness of the tube $\mathcal{T}_{\delta_0}$ and can be seen as follows. Observe that for $\zeta \in M$ the linear terms in $A(\zeta, \cdot)$ vanish and we have $\Re A(\zeta, z) < 0$ for all points $z \in M \backslash \{\zeta\}$ sufficiently close to $\zeta$. Hence for $\delta_0 > 0$ small we can choose $\epsilon > 0$ (depending on $\delta_0$) such that $\Re A(\zeta, z) < 0$ whenever $z, \zeta \in \mathcal{T}_{\delta_0}$ and $\epsilon \leq |\zeta - z| \leq 2\epsilon$. The proof of Theorem 2.4.3 in [HL] (which proceeds by cutting of $\log A$ on $B(\zeta, 2\epsilon) \cap \mathcal{T}_{\delta_0}$ and solving a $\bar{\partial}$-equation on $\mathcal{T}_{\delta_0}$) then gives a globally defined $\Phi$ (and hence $P$).

Let $\Phi, P, A$ and $B$ be as in lemma 3.3, constructed on a fixed tube $\mathcal{T}_{\delta_0}$. $P$ is not quite a Leray map since it does not equal $\overline{\zeta - z}$ near the diagonal, and we shall now modify it suitably on tubes $\mathcal{T}_{\delta}$ for $0 < \delta \leq \delta_0$. Let $0 < c < c' < 1$. Choose a cut-off function $\lambda_\delta$ such that $\lambda_\delta = 1$ in $\mathcal{T}_{c\delta}$ and $\lambda_\delta = 0$ near $\partial \mathcal{T}_{c\delta}$. We may assume that its (real) gradient satisfies $||\nabla \lambda_\delta|| \leq C\delta^{-1}$ for some $C > 0$ independent of $\delta$. We will show that for a suitably chosen function $\phi(\zeta, z)$ on $\overline{\mathcal{T}_{c\delta}} \times \overline{\mathcal{T}_{c\delta}}$, the conditions in Koppelman’s formula (3.6) are satisfied for the pair of domains $\Omega = \mathcal{T}_{\delta}$ and $\Omega' = \mathcal{T}_{c\delta}$ if we define the Leray map $\tilde{P}$ by

$$\tilde{P}(\zeta, z) = (1 - \lambda_\delta(\zeta))\phi(\zeta, z)P(\zeta, z) + \lambda_\delta(\zeta)\overline{\zeta - z},$$

with the corresponding support function given by

$$\Phi = \langle \tilde{P}; \zeta - z \rangle = (1 - \lambda_\delta)\phi \Phi + \lambda_\delta|\zeta - z|^2.$$

We need to find $\phi$ such that $\tilde{\Phi}(\zeta, z) \neq 0$ when $z \in \mathcal{T}_{c\delta}$ and $\zeta \in \overline{\mathcal{T}_{c\delta} \backslash \{z\}}$. When $\zeta \in \overline{\mathcal{T}_{c\delta}}$, we have $\tilde{\Phi}(\zeta, z) = |\zeta - z|^2$, so the condition is satisfied for any choice of $\phi$. Hence it suffices to consider the points $\zeta$ where $\rho(\zeta) > \rho(z)$. Let $\psi: \mathbb{R} \rightarrow [0, 1]$ be a cut-off function such that $\psi(t) = 1$ for $|t| \leq \frac{1}{2}R$ and $\psi(t) = 0$ for $|t| \geq \frac{3}{2}R$. Set

$$\phi(\zeta, z) = \psi(|\zeta - z|)B(\zeta, z)^{-1} + (1 - \psi(|\zeta - z|))\overline{\Phi(\zeta, z)},$$

where $\overline{\Phi(\zeta, z)}$ is the conjugate of $\Phi(\zeta, z)$.
where $B$ is as in lemma 3.3. Then $\phi \Phi = \psi A + (1 - \psi) |\Phi|^2$ (since $B^{-1} \Phi = A$), and we have the following estimates for the real part $\theta(\zeta, z) := \text{Re}\phi \Phi(\zeta, z)$ when $\theta(\zeta) > \theta(z)$:

- when $|\zeta - z| \leq \frac{1}{2} R$, $\theta = \text{Re} A \geq C|\zeta - z|^2$,
- when $\frac{1}{2} R \leq |\zeta - z| \leq R$, $\theta = \psi \text{Re} A + (1 - \psi) |\Phi|^2 \geq \psi C |\zeta - z|^2 + (1 - \psi) C^2 > 0$,
- when $|\zeta - z| > R$, $\theta = |\Phi|^2 \geq C^2$.

This verifies the required properties, and hence (3.6) is valid when $K(\zeta, z)$ and $L(\zeta, z)$ are the kernels generated by the Leray map $P$. For $\zeta$ near $\partial T_b$ we have $P = \phi P$; since $\phi \neq 0$ there, the kernel $L$ is identical to the one generated by the holomorphic Leray map $P$, and hence the first term in (3.6) is zero. This gives us the solution formula (3.7) for the equation $\bar{\partial} v = u$ in $\mathcal{T}_{c, \delta}$. This completes the construction of the kernel for $(0,1)$-forms.

**Proof of the sup-norm estimates.** It suffices to show that the sup-norm estimate holds in our situation when $n \geq 3$. In case $n < 3$ we simply identify $\mathcal{C}^n$ with $\mathcal{C}^n \times \{0\} \subset \mathcal{C}^3$ and extend $f$ independently of the additional variables; the solution to the extended problem will satisfy the estimates, and its restriction to $\mathcal{C}^n$ will be a solution to the original $\bar{\partial}$-problem.

**3.4 Lemma.** The solution $v(z) = \int_{\mathcal{T}_b} K(\zeta, z) \wedge u(\zeta)$ defined above satisfies the estimate $\|v\|_{L^\infty(\mathcal{T}_b)} \leq C \delta \|u\|_{L^\infty(\mathcal{T}_b)}$ when $n \geq 3$.

**Proof.** Let $P^0 = \phi P$, $P^0$ is independent of $\delta$ and $\bar{\partial} P^0 = (1 - \lambda) P^0 + \lambda \zeta \bar{\zeta}$. This gives

$$\bar{\partial} P_j = \bar{\partial} \lambda (\zeta_j - z_j - P_j^0) + (1 - \lambda) \partial P_j^0 + \lambda d \zeta_j = \bar{\partial} \lambda (\zeta_j - z_j - P_j^0) + \eta_j$$

(3.8)

The terms in $K(\zeta, z)$ are of the form $\Phi^{-n} \bar{P}_j \bar{\partial} P[j] \wedge d \zeta$. Since $\bar{\partial} \lambda \wedge \bar{\partial} \lambda = 0$, this is a sum of terms of the following two types:

$$\Phi^{-n} \bar{P}_j \eta[j] \wedge d \zeta \quad \text{and} \quad \Phi^{-n} \bar{P}_j (\bar{\zeta}_k - z_k - P_k^0) \bar{\partial} \lambda \wedge \eta[j, k] \wedge d \zeta.$$

We shall estimate the integrals of these over $\mathcal{T}_b$ when $z \in \mathcal{T}_{c, \delta}$. We have already shown that $\text{Re} \Phi(\zeta, z) \geq C |\zeta - z|^2$. For $|\zeta - z| \leq \frac{1}{2} R$ we have

$$\sum_{j=1}^n P_j^0(\zeta, z)(\zeta_j - z_j) = \langle P^0(\zeta, z), \zeta - z \rangle = A(\zeta, z)$$

$$= 2 \sum_{j=1}^n \frac{\partial \rho(\zeta)}{\partial \zeta_j}(\zeta_j - z_j) + O(|\zeta - z|^2).$$

This implies $P_j^0(\zeta, z) = 2 \frac{\partial \rho(\zeta)}{\partial \zeta_j} + O(|\zeta - z|) = O(\delta + |\zeta - z|)$. By choice of $\lambda$ this gives $(1 - \lambda(\zeta, z)) P_j^0(\zeta, z) = O(|\zeta - z|)$ and therefore $P_j(\zeta, z) = O(|\zeta - z|)$. Since $|\eta_j| \leq C$, we get

$$\Phi^{-n} \bar{P}_j \eta[j] \wedge d \zeta = O(|\zeta - z|^{1 - 2n}),$$

$$\Phi^{-n} \bar{P}_j (\bar{\zeta}_k - z_k - P_k^0) \bar{\partial} \lambda \wedge \eta[j, k] \wedge d \zeta = O(|\zeta - z|^{1 - 2n} + \delta^{-1} |\zeta - z|^{2 - 2n}).$$
which shows that the kernel $K(\zeta, z)$ has a singularity of the same type as the Bochner-Martinelli kernel on the diagonal.

Locally we may straighten $M$, i.e., for each $p \in M$ there is a neighborhood $V_p$ of $p$ and a $C^1$-diffeomorphism $\Psi: \mathcal{U} \to V_p$, where $\mathcal{U}$ is a neighborhood of the origin in $\mathbb{R}^{2n}$, such that $\Psi$ is nearly volume and distance preserving, and $\Psi(V_p \cap \mathbb{R}^m) = V_p \cap M$, where $m$ is the dimension of $M$. We denote the points in $\mathbb{R}^{2n}$ by $(u', u'') \in \mathbb{R}^m \times \mathbb{R}^{2n-m}$. By compactness we may assume that $\mathcal{T}_\delta$ is covered by a finite number (independent of $\delta$) of sets

$$K_j^\delta = \Psi_j(\{(u', u''); |u'| \leq a, |u''| \leq \delta\})$$

for some constant $a$. We keep the notation $\zeta$ and $z$ for the points in the new coordinates also. We then have the estimate ($\zeta = (u', u'')$):

$$\left| \int_{K_j^\delta} K(\zeta, z) \wedge u(\zeta) \right| \leq C \|u\|_{L^\infty(\mathcal{T}_\delta)} \int_{|u'| \leq a, |u''| \leq \delta} (|\zeta - z|^{1-2n} + \delta^{-1} |\zeta - z|^{2-2n}) dV(\zeta)$$

$$\leq C \|u\|_{L^\infty(\mathcal{T}_\delta)} \int_{|u'| \leq a, |u''| \leq \delta} (|\zeta|^{1-2n} + \delta^{-1} |\zeta|^{2-2n}) dV(\zeta).$$

For $m < t < 2n$ we estimate these integrals as follows:

$$\int_{|u'| \leq a, |u''| \leq \delta} \frac{1}{|\zeta|^t} \leq C \left( \int_0^{\sqrt{2\delta}} \frac{r^{2n-1} dr}{r^t} + \int_{\delta}^{a} \frac{\delta^{2n-m} r^{m-1} dr}{r^t} \right) \leq C\delta^{2n-t}. \quad (3.9)$$

Hence

$$\left| \int_{K_j^\delta} K(\zeta, z) \wedge u(\zeta) \right| \leq C \|u\|_{L^\infty(\mathcal{T}_\delta)} (\delta + \delta^{-1} \delta^2) = 2C \|u\|_{L^\infty(\mathcal{T}_\delta)}$$

when $2n - 2 > m$. Since $m \leq n$, this holds for $n > 2$. ♠

Construction of the kernel for forms of higher degree. We consider the form

$$K(\zeta, z) = c_n \Phi(\zeta, z)^{-n} \sum_{j=1}^{n} (-1)^{j-1} \bar{P}_j \partial \bar{P}_j^* \wedge d(\zeta - z)$$

on $\mathcal{T}_\delta \times \mathcal{T}_\delta$, where $\partial$ is now taken with respect to both $\zeta$ and $z$. We decompose

$$K(\zeta, z) = \sum_{p \leq n} \sum_{q \leq n-1} K_{p,q}(\zeta, z),$$

where $K_{p,q}$ has bidegree $(p, q)$ with respect to $z$ and $(n-p, n-q-1)$ with respect to $\zeta$. When $q > 0$, $K_{p,q}(\zeta, z) = 0$ when $z \in \mathcal{T}_\delta$ and $\zeta$ is near $\partial \mathcal{T}_\delta$. (Recall that $K(\zeta, z) = K^\delta(\zeta, z)$ depends on $\delta$ via the cutoff function $\lambda_\delta$.) It follows that the $(p, q-1)$-form

$$v(z) = \int_{\mathcal{T}_\delta} K_{p,q-1}(\zeta, z) \wedge u(\zeta) = (-1)^{p+q} \int_{\mathcal{T}_\delta} u(\zeta) \wedge K_{p,q-1}(\zeta, z)$$
solves $\bar{\partial}v = u$ in $T_{\varepsilon, \delta}$ for each $\bar{\partial}$-closed $(p, q)$-form $u$ in $T_{\varepsilon, \delta}$, $q > 0$. The precise meaning of the integral is as follows. Write

$$K_{p, q-1}(\zeta, z) = \sum_{|I|=p} \sum_{|J|=q-1} k_{I, J}(\zeta, z) dz^I \wedge d\bar{z}^J,$$

where $k_{I, J}(\zeta, z)$ is an $(n - p, n - q)$-form in $\zeta \in T_{\varepsilon, \delta}$ depending smoothly on $z \in T_{\varepsilon, \delta}$. Then

$$v(z) = \sum_{|I|=p} \sum_{|J|=q-1} (-1)^{p+q} \left( \int_{T_{\varepsilon, \delta}} u(\zeta) \wedge k_{I, J}(\zeta, z) \right) dz^I \wedge d\bar{z}^J.$$ 

This completes the construction of the kernel. The reader may find some additional references and historical remarks about the solution formula at the end of this section.

Before proceeding we make the following elementary

**Geometric observations.** Let $M$ be a compact $m$-dimensional $C^1$-submanifold of $\mathbb{R}^N$. There exists a constant $B > 0$ such that, if $z_0, z_1 \in T_{\varepsilon, \delta}(M)$ for sufficiently small $\delta$, then $z_0$ and $z_1$ may be joined by a path in $T_{\varepsilon, \delta}(M)$ of length no more than $B|z_1 - z_0|$. This is due to the fact that the tubes may be locally straightened, in a uniform way, to tubes around $\mathbb{R}^m \times \{0\}$ in $\mathbb{R}^N$.

From this we get the following: If $u \in C^1(T_{\varepsilon, \delta}(M), \|u\|_{L^\infty(T_{\varepsilon, \delta})} \leq A$, $\|u\|_{C^1(T_{\varepsilon, \delta})} \leq At^{-1}$ for $t \leq 1$ and $0 < s < 1$, then $|u|_{s, \delta} \leq \max(2, B)At^{-s}$. We see this as follows: If $|h| \leq t$, we can integrate $Du$ from $z$ to $z + h$ to get $|u(z + h) - u(z)||h||^{-s} \leq BAt^{-1}|h|^{1-s} \leq BAt^{-s}$. If $|h| \geq t$, the triangle inequality gives $|u(z + h) - u(z)||h||^{-s} \leq 2At^{-s}$.

We also have a corresponding result for compact manifolds $M$: if $\|u\|_{C^r(M)} \leq A$ and $\|D^j u\|_{C^{r+j}(M)} \leq At^{-1}$ for $t \geq 0$, then $\|u\|_{C^{r+j}(M)} \leq CAt^{-s}$, where $C$ is a constant independent of $u$.

**Proof of the estimates for forms of higher degree.** The proof of the sup-norm estimate, which we gave for $(0, 1)$-forms, carries over almost verbatim to the general case. However, lemma 3.2 almost certainly fails, at least for the solutions constructed here, and we must proceed differently to estimate the derivatives.

With $c_0 = c' - c$ we introduce smooth cut-off functions $\chi_{\delta} \in C_0^\infty(B(0, c_0\delta))$ with $\chi_{\delta}(w) = 1$ when $|w| < c_0\delta/2$ and $|\partial^\alpha \chi_{\delta}| \leq C_\alpha \delta^{-|\alpha|}$. Then we decompose $v$ as $v' + v''$, with

$$v'(z) = \int_{T_{\varepsilon, \delta}} \chi_{\delta}(\zeta - z)K_{p, q-1}(\zeta, z) \wedge u(\zeta),$$

$$v''(z) = \int_{T_{\varepsilon, \delta}} (1 - \chi_{\delta}(\zeta - z))K_{p, q-1}(\zeta, z) \wedge u(\zeta),$$

and estimate each summand separately.

Recall that when $z \in T_{\varepsilon, \delta}$ and $|\zeta - z| \leq c_0\delta$, $K(\zeta, z)$ equals the Bochner-Martinelli kernel. Thus $v'(z)$ is obtained for $z \in T_{\varepsilon, \delta}$ by applying a convolution operator to $u$; hence

$$\partial^\alpha v'(z) = \int_{T_{\varepsilon, \delta}} \chi_{\delta}(\zeta - z)K_{p, q-1}(\zeta, z) \wedge \partial^\alpha u(\zeta).$$
Thus the components of $\partial^{\alpha}u'(z)$ are linear combinations of terms $h(z) = (k \ast g)(z)$, where $k(w) = \chi_{\delta}(w)\overline{\partial}_{j}|w|^{-2n}$ and $g$ is a component of $\partial^{\alpha}u$. Since $|k(w)| \leq |w|^{-2n}$ and $k$ is supported by $B(0, c_0 \delta)$, an obvious estimate gives $|h(z)| \leq C\delta \|g\|_{\infty}$, so

$$\|\partial^{\alpha}u'\|_{L^{\infty}(T_{c\delta})} \leq C\delta \|\partial^{\alpha}u\|_{L^{\infty}(T_{c\delta})}.$$  

To estimate the finer norms of $h$ we introduce the auxiliary kernels

$$k_t(w) = \chi_{\delta}(w)\overline{\partial}_{j}(t^2 + |w|^2)^{-n}; \quad t > 0.$$  

This is a smooth function of $(t, z)$ satisfying $|k_t(z)| \leq |k(z)|$ and $\lim_{t \to 0} k_t(z) = k(z)$. Since each $k_t$ has compact support, it follows that $\int \partial^{\alpha}D^\beta h_t(z) dV(w) = 0$ for every $(t, z)$-derivative $D^\beta$. Thus, setting $h_t(z) = (k_t \ast g)(z)$, we see that

$$D^\beta \partial^{\alpha}h_t(z) = \int_{T_{c\delta}} \partial_j D^\beta k_t(w)(g(z - w) - g(z)) dV(w).$$  

Observing that $|\partial^\gamma\chi_{\delta}(w)| \leq C_{\gamma}|w|^{-1}$ on $\text{supp} \chi_{\delta}$, a simple calculation gives

$$|D^\beta \partial^{\alpha}h_t(w)| \leq C_{\beta}|w|^{-1}\delta(t + |w|)^{-2n}. $$

Assume that $\partial^{\alpha}u \in \Lambda^s(T_{c\delta})$ for some $s \in (0, 1)$. We have $g \in \Lambda^s(T_{c\delta})$, and for $t > 0$ we can estimate in polar coordinates:

$$|\partial_j h_t(z)| \leq C |g|_{s, \delta} \int_{|w| < c_0 \delta} (t + |w|)^{s-2n} dV(w)$$

$$\leq C |g|_{s, \delta} \int_{0}^{c_0 \delta} r^{s-1} dr = C_s \delta^s |g|_{s, \delta}.$$  

For the first order derivatives with respect to $(t, z)$ we get in the same way:

$$|D^\beta \partial_j h_t(z)| \leq C |g|_{s, \delta} \int_{|w| < c_0 \delta} |w|^{s-1}(t + |w|)^{-2n} dV(w)$$

$$\leq C |g|_{s, \delta} \int_{0}^{c_0 \delta} (t + r)^{s-2} dr \leq C' t^{s-1} |g|_{s, \delta}. \quad (3.10)$$  

By the dominated convergence theorem we have $h_t(z) \to h(z)$ and

$$\partial_j h_t(z) \to h(j)(z) = \int_{T_{c\delta}} \partial_j k(w)(g(z - w) - g(z)) dV(w)$$  

as $t \to 0$. We also have

$$|\partial_j h_t(z) - h(j)(z)| \leq \int_{0}^{t} |\partial_j h_{\tau}(z)| d\tau \leq C |g|_{s, \delta} t^s;$$

hence the convergence of the derivatives is uniform and therefore $h(j)(z) = \partial_j h(z)$. Thus $|\partial_j h_t(z)| \leq C_s \delta^s |g|_{s, \delta}$, and we conclude that

$$\|\partial_j \partial^{\alpha}u'\|_{L^{\infty}(T_{c\delta})} \leq C_s \delta^s |\partial^{\alpha}u|_{s, \delta}.$$  

We have also shown that

$$\partial_j \partial^{\alpha}u'(z) = \int_{T_{c\delta}} \partial_j \left(\chi_{\delta}((\zeta - z)K_{p, q - 1}(\zeta, z)) \wedge (\partial^{\alpha}u(\zeta) - (\partial^{\alpha}u)_z)\right).$$

In order to estimate the $\Lambda^s$-norm of $\partial_j \partial^{\alpha}u'$ we need the following standard
Lemma. Let \( \phi \in C(T_\delta) \) have an extension \( \bar{\phi} \in C^1(\mathbb{R}^+ \times T_\delta) \) satisfying \(|D\bar{\phi}(t, z)| \leq At^{s-1}\) for some \(0 < s < 1\). Then \( \phi \in \Lambda^s(T_\delta) \) and \(|\phi|_{s, \delta} \leq (B + 2/s)A\).

This is just a slight modification of Proposition 2 in the Appendix 1 of [HL]. Applying this to \( \phi = \partial_j h \) and \( \phi(t, z) = \partial_j h_t(z) \), (3.10) gives \(|\partial_j h|_{s, \delta} \leq C_s(B + 2/s)|g|_{s, \delta} \). Thus

\[
|\partial_j^2 \partial^\alpha v|_{s, \delta} \leq C_s|\partial^\alpha u|_{s, \delta}.
\]

In order to study \( v'' \) we set \( K''_{p, q-1}(\zeta, z) = (1 - \chi_\delta(\zeta - z))K_{p, q-1}(\zeta, z) \) on \( T_\delta \times T_{c\delta} \). This kernel has continuous \( z \)-derivatives of all orders, and it equals zero when \(|\zeta - z| \leq c_0\delta/2\). It follows that \( v'' \) is a smooth form with

\[
\partial^\alpha v''(z) = \int_{T_\delta} \partial^\alpha_z K''_{p, q-1}(\zeta, z) \wedge u(\zeta).
\]

We recall the formula (3.8) and point out that \(|\partial\lambda_\delta| = O(\delta^{-1}), |\phi(\zeta, z)| \geq C|\zeta - z|^2\) on \( T_\delta \times T_{c\delta} \), and the quantities \(|D_z \phi(\zeta, z)|, |\eta_j|\) and \(|P^0_j|\) are all bounded by \( C|\zeta - z|\), while their derivatives with respect to \( z \) are bounded independently of \( \delta \) (since \( \lambda_\delta \) is independent of \( \delta \)).

An induction on \(|\alpha|\) shows that the components of \( \partial^\alpha_z K''_{p, q-1}(\zeta, z) \) are linear combinations of terms of the type

\[
\bar{\phi}^{-n-k} \partial^\beta_z (1 - \chi_d(\zeta - z)) a_0(\zeta, z) \cdots a_t(\zeta, z)
\]

with \( \beta \leq \alpha, k \leq |\alpha - \beta| \) and \( t \geq 2k + 1 - |\alpha - \beta| \), and of terms of the type

\[
\bar{\phi}^{-n-k} \frac{\partial \lambda_\delta(\zeta)}{\partial \zeta^t} a_0(\zeta, z) \cdots a_t(\zeta, z)
\]

with \( k \leq |\alpha| \) and \( t \geq 2k + 2 - |\alpha| \), where the \( a_j(\zeta, z) \) have continuous \( z \)-derivatives of all orders that have upper bounds independent of \( \delta \), and \(|a_j(\zeta, z)| \leq C|\zeta - z|\) when \( t > 0 \) and \( 1 \leq j \leq t \). Since \(|\zeta - z| > c_0\delta/2\) when \( K''_{p, q-1} \neq 0 \), it follows easily that

\[
|\partial^\alpha_z K''_{p, q-1}(\zeta, z)| \leq C_\alpha \delta^{-1} |\zeta - z|^{2-2n-|\alpha|}.
\]

Thus

\[
|\partial^\alpha v''(z)| \leq C_\alpha \delta^{-1} \|u\|_{L^\infty(T_\delta)} \int_{T_\delta \setminus B(z, c_0\delta/2)} |\zeta - z|^{2-2n-|\alpha|} dV(\zeta)
\]

\[
\leq C_\alpha \delta^{-1-|\alpha|} \|u\|_{L^\infty(T_\delta)}
\]

for \( z \in T_{c\delta}, \alpha \in \mathbb{Z}_+^n \). The last estimate follows for \(|\alpha| > 2, |\alpha| = 2 \) and \(|\alpha| = 1 \), respectively, from the following three integral estimates:

\[
\int_{|\zeta - z| > \delta} |\zeta - z|^{-2n-t} dV(\zeta) = C_\delta \delta^{-t}, \quad t > 0,
\]

(3.11)

\[
\int_{T_{c\delta} \setminus B(z, c_1\delta)} |\zeta - z|^{-2n} dV(\zeta) \leq C(c_1), \quad z \in T_\delta,
\]

(3.12)

\[
\int_{T_\delta} |\zeta - z|^{s-2n} dV(\zeta) \leq C_s \delta^s; \quad 0 < s < 2n - m.
\]

(3.13)
(3.11) follows immediately by a change of variable. (3.12) is proved exactly like (3.9); in
the sum in the middle of (3.9) the first integral has lower limit $c_1 \delta$ instead of 0. Finally
(3.13) follows by setting $t = 2n - s$ in (3.9).

Using the geometric observation following the construction of the kernel we have
\[ \| \partial^\alpha v^\prime \|_{L^\infty(\mathcal{T}_\delta)} \leq C_{\alpha,s} \delta^{-1-|\alpha|-s} \| u \|_{L^\infty(\mathcal{T}_\delta)}. \]
This completes the proof of the H"older estimates in theorem 3.1 for the case $0 < s < 1$. The proof for $s = 1$ goes along the same lines, with certain small modifications; since that case will not be used in the paper, we omit the details.

\*\*\*

Remarks on constructions of kernels. The first integral kernel operators with holomor-
phic kernels, solving the $\bar{\partial}$-equation on strongly pseudoconvex domains in $\mathbb{C}^n$, have been
constructed by Henkin and, independently, R. de Arellano (see the references in [HL]).
Henkin's approach is to patch the Bochner-Martinelli and Leray kernels on the boundary
$\partial \Omega$. Our patching of the two kernels (by first multiplying by $\phi$) is the same as in "{O}vrelid
[O1, O2]. The whole construction is similar to the one by Harvey and Wells [HaW].

It seems that the first really precise $L^\infty$ and $\mathcal{C}^k$-estimates for the $\bar{\partial}$-equation in thin
tubes around a totally real submanifold $M \subset \mathbb{C}^n$, proved by means of integral solution
operators, are due to Harvey and Wells [HaW] in 1972. A little later Range and Siu
[RS] (1974) used a more refined kernel construction to prove estimates for the highest
order derivatives of their solution on $M$ and deduced $\mathcal{C}^k$-approximation of $\mathcal{C}^k$-functions
on a $\mathcal{C}^k$-submanifold $M \subset \mathbb{C}^n$ by holomorphic functions, a case left open in [HaW]. In
fact this approximation problem has been one of the original motivations in proving such
estimates. Later on this approximation has been accomplished more efficiently, and in
greater generality, by Baouendi and Treves [BT1, BT2] by using the convolution with the
complex Gaussian kernel. This latter method does not seem to give the approximation of
diffeomorphisms obtained in this paper because we must work in tubular neighborhoods
and not solely on the submanifold.

As said earlier our construction of the kernel in this paper is close to [HaW], and our
main contribution is the way we estimate the solutions. We find it quite striking that
this simple and seemingly crude construction of the kernel gives rise to results that are
essentially optimal for the applications to mappings presented in this paper. For the benefit
of the reader we have given a fairly self contained presentation based on the text [HL].
Another closely related paper is [BB] where Bruna and Burgués approximate $\bar{\partial}$-closed jets
on a totally real set $X$ in H"older norms by functions holomorphic in a neighborhood of $X$.
It seems likely that their method, making use of weighted integral kernels of Anderson and
Berndtsson type [AB], may also be used to prove our results. However, we believe that our
approach is simpler and more elementary. Our results, suitably reformulated, may also be
proved for neighborhoods of totally real sets.

\&4. Proof of theorems 1.2 and 1.3.

Proof of theorem 1.2. We consider first the case $\dim M_0 = \dim M_1 = n$. Let $d(z)$ denote
the Euclidean distance of $z$ to $M_0$, and let $\mathcal{T}_\delta$ (resp. $\mathcal{T}_\delta'$) denote the open tube of radius $\delta$
around $M_0$ (resp. around $M_1$). The $C^k$-diffeomorphism $f: M_0 \to M_1$ can be extended to a $C^k$-map on $\mathbb{C}^n$, still denoted $f$, which is $\partial\overline{\partial}$-flat to order $k$ at $M_0$:

$$|\partial^\alpha (\partial \overline{\partial} f)(z)| = o(d(z)^{k-1-|\alpha|}); \quad 0 \leq |\alpha| \leq k - 1.$$ 

In particular, the derivative $Df(z)$ is a non-degenerate $\mathbb{C}$-linear map at each point $z \in M_0$ (the complexification of $df_z: T_zM_0 \to T_{f(z)}M_1$), and hence $f$ is a $C^k$ diffeomorphism in some neighborhood of $M_0$ in $\mathbb{C}^n$. The $(0,1)$-form $u = \partial \overline{\partial} f$ of class $C^{k-1}$ satisfies $\partial \overline{\partial} u = 0$ and

$$||\partial^\alpha u||_{L^\infty (\mathcal{T}_\delta)} = o(\delta^{k-1-|\alpha|}); \quad 0 \leq |\alpha| \leq k - 1$$

as $\delta \to 0$. Applying theorem 3.1 (specifically the estimates (3.1), with $l = k - 1 \geq 0$ and a fixed constant $0 < c < 1$), we get for each sufficiently small $\delta > 0$ a solution $v_\delta$ to $\partial \overline{\partial} v_\delta = u$ in $\mathcal{T}_\delta$ satisfying the following estimates:

$$||\partial^\alpha v_\delta||_{L^\infty (\mathcal{T}_\delta)} \leq C \left( \delta ||\partial^\alpha u||_{L^\infty (\mathcal{T}_\delta)} + \delta^{1-|\alpha|} ||u||_{L^\infty (\mathcal{T}_\delta)} \right) \leq C \left( \delta o(\delta^{k-1-|\alpha|}) + \delta^{1-|\alpha|} o(\delta^{k-1}) \right) = o(\delta^{k-1-|\alpha|}); \quad |\alpha| \leq k - 1.$$

Moreover, since $\partial \overline{\partial} v = u$ has a solution of class $C^{l+1} = C^k$, namely $f$, we can choose $v_\delta$ which in addition satisfies the estimates (3.2) for the derivatives of top order $k$:

$$||\partial^\alpha v_\delta||_{L^\infty (\mathcal{T}_\delta)} \leq C \left( \omega_k(f; \delta) + \delta^{-k} ||\partial \overline{\partial} f||_{L^\infty (\mathcal{T}_\delta)} \right) = o(1); \quad |\alpha| = k.$$

Here $\omega_k(f; \delta)$ denotes the modulus of continuity of the $k$-th order derivatives of $f$. Set $F_\delta = f - v_\delta$ in $\mathcal{T}_\delta$. Then $\partial \overline{\partial} F_\delta = 0$ and the estimates on $v_\delta$ imply

$$||F_\delta - f||_{C^r (\mathcal{T}_\delta)} = ||v_\delta||_{C^r (\mathcal{T}_\delta)} = o(\delta^{k-r}); \quad 0 \leq r \leq k$$

which gives the first estimate in (1.2). It remains to prove that $F_\delta$ is biholomorphic and satisfies the inverse estimates in (1.2) for all sufficiently small $\delta > 0$. To simplify the notation we replace $\delta$ by $c\delta / 2$, so that $F_\delta$ is holomorphic in the tube $\mathcal{T}_{2\delta}$ and it satisfies

$$||F_\delta - f||_{C^r (\mathcal{T}_{2\delta})} = o(\delta^{k-r}); \quad 0 \leq r \leq k$$

as $\delta \to 0$. Since $f$ is a diffeomorphism near $M_0$, so is any sufficiently close $C^1$ approximation of $f$; hence (4.1) with $r = 1$ implies that for $\delta > 0$ sufficiently small, say $0 < \delta \leq \delta_0 \leq 1$, the map $F_\delta$ is diffeomorphic (and hence biholomorphic) in $\mathcal{T}_{2\delta}$. Decreasing $\delta_0$ if necessary, there is a number $a > 0$ such that

$$|f(z) - f(z')| \geq 2a|z - z'|; \quad z, z' \in \mathcal{T}_{\delta_0}.$$ 

Since $f(M_0) = M_1$, the above implies that $f(\mathcal{T}_{\delta})$ contains the tube $\mathcal{T}_{2a\delta}$.
Fix an $\epsilon > 0$. By (4.1), applied with $r = 0$, we get a constant $\delta_1 = \delta_1(\epsilon)$, with $0 < \delta_1 \leq \delta_0$, such that $||F_\delta - f||_{L^\infty(T_2^s)} < a\epsilon \delta^k$ for $0 < \delta \leq \delta_1$. Fix a point $z \in T_\delta$ and let $w = f(z)$. For each $z'$ with $|z' - z| = \epsilon \delta^k$ we have

$$|F_\delta(z') - w| = |(F_\delta(z') - f(z')) + (f(z') - f(z))|$$

$$\geq |f(z') - f(z)| - |F_\delta(z') - f(z')|$$

$$\geq 2a\epsilon \delta^k - a\epsilon \delta^k = a\epsilon \delta^k.$$  

This means that the image by $F_\delta$ of the sphere $S = \{z': |z' - z| = \epsilon \delta^k\}$ is a hypersurface containing the ball $B(w; a\epsilon \delta^k) = \{w': |w' - w| < a\epsilon \delta^k\}$ in the bounded component of its complement. By degree theory the $F_\delta$-image of the ball $B(z; \epsilon \delta^k)$ contains the ball $B(w; a\epsilon \delta^k)$. Hence there is a point $\zeta \in B(z; \epsilon \delta^k)$ such that $F_\delta(\zeta) = w = f(z)$, and we have $|F_\delta^{-1}(w) - f^{-1}(w)| = |z - z| < \epsilon \delta^k$. Since this applies to any point $w \in T_{2a\delta}'$, we conclude that $F_\delta(T_\delta) \supset T_{2a\delta}'$ and

$$||F_\delta^{-1} - f^{-1}||_{L^\infty(T_{2a\delta})} \leq a\epsilon \delta^k; \quad 0 < \delta \leq \delta_1(\epsilon). \quad (4.2)$$

Since $\epsilon > 0$ was arbitrary, this gives the inverse estimate in (1.2) for $r = 0$.

We proceed to estimate the derivatives of the inverse maps. Denote by $||A||$ the spectral norm of a linear map $A \in GL(R, 2n)$. Note that $DF^{-1}(w) = Df(z)^{-1}$ where $w = f(z)$. Fix a point $w \in T_{2a\delta}'$ and let $z = f^{-1}(w)$, $z_\delta = F_\delta^{-1}(w)$ (these are points in $T_{2\delta}$). By (4.2) we have $|z - z_\delta| \leq \epsilon \delta^k$. Writing $A = Df(z)$, $B = DF_\delta(z_\delta)$, we get

$$||DF_\delta^{-1}(w) - Df^{-1}(w)|| = ||A^{-1} - B^{-1}||$$

$$= ||A^{-1}(B - A)B^{-1}||$$

$$\leq ||A^{-1}|| \cdot ||A - B|| \cdot ||B^{-1}||.$$  

Since $f$ is a diffeomorphism and $F_\delta$ is $C^1$-close to $f$, the eigenvalues of $A$ and $B$ are uniformly bounded away from zero, and this gives a uniform estimate on $||A^{-1}||$ and $||B^{-1}||$ (independent of $\delta$). The middle term is

$$||A - B|| = ||Df(z) - DF_\delta(z_\delta)|| \leq ||Df(z) - Df(z_\delta)|| + ||Df(z_\delta) - DF_\delta(z_\delta)||.$$  

The second term on the right hand side is of size $o(\delta^{k-1})$ according to (4.1). As $\delta \to 0$, we have $z_\delta \to z$, and hence the first term on the right hand side goes to zero (by continuity of $Df$). Hence $\sup\{|||DF_\delta^{-1}(w) - Df^{-1}(w)||; w \in T_{2\delta}'\}$ goes to zero as $\delta \to 0$. This completes the proof when $k = 1$. If $k > 1$, we can further estimate $||Df(z) - Df(z_\delta)|| \leq C|z - z_\delta| \leq C\epsilon \delta^k$, where $C$ is an upper bound for the second derivatives of $f$. This gives

$$\sup\{|||DF_\delta^{-1}(w) - Df^{-1}(w)||; w \in T_{2\delta}'\} = o(\delta^{k-1})$$

as required by (1.2) for derivatives or order $r = 1$. To get the estimates (1.2) for the higher derivatives of $F_\delta^{-1} - f^{-1}$ we may apply the same method to the tangent map, i.e., the induced map on tangent bundles over the tubes which equals the derivative of the
given map on each tangent space. We leave out the details. This proves theorem 1.2 when \( \dim M_0 = n \).

Suppose now that \( m = \dim M_0 < n \). We are assuming that there is an isomorphism \( \phi: \nu_0 \to \nu_1 \) of the complex normal bundles \( \nu_0 \to M_0 \) resp. \( \nu_1 \to M_1 \) over \( f \); by approximation we may assume that \( \phi \) is of class \( C^{k-1} \). For each \( z \in M_0 \) we have \( T_z C^n = T_z^C M_0 \oplus \nu_{0,z} \). Let \( l_z \) be the \( \mathbb{C} \)-linear map on \( C^n \) which is uniquely defined by taking \( l_z = df_z \) on \( T_z^C M_0 \) and \( l_z = \phi_{0,z} \) on \( \nu_{0,z} \). Clearly \( l_z \in GL(n, \mathbb{C}) \) for each \( z \in M \). Applying lemma 2.6 we obtain a \( C^k \)-extension \( f \) of \( f \) which is \( \overline{\partial} \)-flat on \( M_0 \). Now the proof may proceed exactly as before. This proves theorem 1.2. ♣

Remarks. 1. If \( f_0: M_0 \to M_1 \) is a real-analytic diffeomorphism and if the complex normal bundles to \( M_0 \) resp. \( M_1 \) are isomorphic over \( f \) then \( f \) extends to a biholomorphic map \( F \) from neighborhood of \( M_0 \) onto a neighborhood of \( M_1 \). We see this as follows. Let \( \phi: \nu_0 \to \nu_1 \) be the continuous isomorphism (over \( f \)) of the complex normal bundles to \( M_0 \) resp. \( M_1 \). There exist complexifications \( \tilde{M}_i \subset C^n \) of \( M_i \) \( (i = 0, 1) \) such that \( f \) extends to a biholomorphic map \( \tilde{f}: \tilde{M}_0 \to \tilde{M}_1 \) and such that the complex normal bundles \( \nu_i \to M_i \) extend to holomorphic vector bundles \( \tilde{\nu}_i \to \tilde{M}_i \). We define a continuous map \( \psi: M_0 \to GL(n, \mathbb{C}) \) by \( \psi(z) = df_z \oplus \phi_z \). Since \( M_0 \subset \tilde{M}_0 \) is totally real, \( \psi \) may be approximated by a holomorphic map \( \tilde{\psi}: \tilde{M}_0 \to GL(n, \mathbb{C}) \). We now define \( \tilde{\phi}: \nu_0 \to \tilde{M}_1 \times C^n \) by \( \tilde{\phi}(v) = (\tilde{f}(z), \tilde{\psi}(v)) \). Clearly \( \tilde{\phi} \) is a holomorphic vector bundle isomorphism between \( \nu_0 \) and a holomorphic sub-bundle \( \tilde{\nu}_2 \subset \tilde{M}_1 \times C^n \) which is an approximation of \( \tilde{\nu}_1 \). In particular, \( \phi \) is a biholomorphic map between neighborhoods \( V_i \) of the zero sections of \( \nu_i \). By the tubular neighborhood theorem these neighborhoods map biholomorphically onto neighborhoods of \( M_0 \) resp. \( M_1 \) under the projection maps. This gives the desired biholomorphic extension of \( f \).

2. If instead of theorem 3.1 we use Hörmander’s \( L^2 \)-estimates when solving \( \overline{\partial} u = u(= \overline{\partial} f) \) in \( T_o \), the resulting holomorphic maps \( F_\delta = f - u_\delta \) can be shown to satisfy the weaker estimate \( \|F_\delta\|_{C^r(M_0)} = o(\delta^{k-r-l}) \) for \( 0 \leq r \leq l \), where \( l \) is the smallest integer larger than \( \frac{1}{2} \dim M_0 \). This approach had been used in [FL].

Proof of theorem 1.3. The proof can be obtained by repeating the proof of theorem 1.1 in [FL] (or its more technical version, theorem 2.1 in [FL]), except that one applies theorem 3.1 above whenever solving a \( \overline{\partial} \)-equation. This gives the improved estimates in (1.3) with no loss of derivatives. We leave out the details. ♣

A correction to [FL]. We take this opportunity to correct an error in the proof of Lemma 4.1 in [FL]. The equation numbers below refer to that paper. The lemma is correct as stated, but the proof of the estimate (4.5) is not correct. Using the notation of that proof, we have the higher variational equations

\[
\frac{\partial}{\partial t} D^p \phi_t(x) = DX_t(\phi_t(x)) \circ D^p \phi_t(x) + H^p_X(t, x)
\]

for \( p \leq k \), where \( D^p f \) denotes the \( p \)-th order derivative of a map \( f: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \), so \( D^p f \in L^p(\mathbb{R}^n, \mathbb{R}^n) \). \( H^p_X(t, x) \) is a sum of terms involving derivatives of the vector field \( X_t \)

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and derivatives of order less than \(p\) of the flow \(\phi_t\), and \(H^1_{X} = 0\). We use the same notation for \(Y^t\) and its flow \(\psi^t\).

Choose unit vectors \(v_1, \ldots, v_p \in \mathbb{R}^n\) and set \(y(t) = [DP\phi_t(x) - DP\psi^t(x)](v_1, \ldots, v_p)\). It will be sufficient to show that \(\|y(t)\| = o(e^{k-p})\), uniformly for \(0 \leq t \leq t_0, x \in K(\epsilon)\) and unit vectors \(v_1, \ldots, v_p\). \(y(t)\) satisfies the differential equation

\[
y'(t) = D\psi_t^*(\psi^t(x)) \cdot y(t) + (DX_t(\phi_t(x)) - D\psi_t^*(\psi^t(x)) \circ DP\phi_t(x)(v_1, \ldots, v_p) \\
+ (H_X^p(t, x) - H_{Y^t}(t, x))(v_1, \ldots, v_p).
\]

This is a linear system \(y' = A(t) \circ y + b(t), y \in \mathbb{R}^n\). Suppose the matrix norms satisfy \(\|A(t)\| \leq A \) and \(\|b(t)\| \leq b\) for \(t \in [0, t_0]\). The function \(u(t) = \|y(t)\|\) is differentiable outside the zeroes of \(u\), with \(u'(t) = y'(t) \cdot y(t)/\|y(t)\| \leq \|y'(t)\|\), so \(u'(t) \leq Au(t) + b\) outside the zeroes of \(u\). Since \(\phi_0 = \psi_0 = Id\), we have \(y(0) = 0\). We shall first show that \(u(t) \leq \frac{b}{A}(e^{At} - 1)\) for \(t \in [0, t_0]\). If \(u(t) = 0\), there is nothing to prove. If not, let \(t_1\) be the largest zero of \(u\) on \([0, t]\). Thus \(u'(s) \leq Au(s) + b\) for \(s \in (t_1, t]\). Setting \(v(s) = u(s)e^{-As}\) we get \(v'(s) \leq be^{-As}\) for \(s \in (t_1, t]\). Integration from \(t_1\) to \(t\) gives \(v(t) \leq \frac{b}{A}(e^{-At} - e^{-At_1})\). Thus \(u(t) \leq \frac{b}{A}(e^{At} - 1)\) for \(t \in [0, t_0]\).

In our situation, the matrix norm of \(A(t) = D\psi_t^*(\psi^t(x))\) is bounded independently of \(\epsilon > 0, x \in K(\epsilon)\) and \(t\), by (4.4). It is therefore sufficient to prove that \(b = o(e^{k-p})\), uniformly in \(t\) and unit vectors \(v_1, \ldots, v_p\). It is shown in [FL] that the matrix norm \(\|DX_t(\phi_t(x)) - D\psi_t^*(\psi^t(x))\|_{L^\infty(K(\epsilon))} = O(e^{k-1})\). Since the flow \(\phi_t(x)\) is of class \(C^k\), it follows that the matrix norm \(\|DP\phi_t(x)\|\) is uniformly bounded for \(x \in K(\epsilon)\) and \(t \in [0, t_0]\). Applying (4.4) and (4.5) inductively as in [FL] we obtain \(\|H_x^p - H_{Y^t}\|_{L^\infty(K(\epsilon))} = O(e^{k-p})\), uniformly in \(t\), which proves the claim.

\&5. Solving the equation \(dv = u\) for holomorphic forms in tubes.

Let \(d\) denote the exterior derivative. In this section we solve the equation \(dv = u\) with sup-norm estimates for holomorphic forms in tubes \(T_\delta = \mathbb{T}_\delta M\) around totally real submanifolds \(M \subset \mathbb{C}^n\). We denote by \(\Lambda^s\) the Hölder spaces as in sect. 3 above. We first state our main result for closed submanifolds; for an extension to compact submanifolds with boundary see remark 3 following theorem 5.1.

5.1 Theorem. Let \(i: M \hookrightarrow \mathbb{C}^n\) denote the inclusion of a closed, \(m\)-dimensional, totally real submanifold of class \(C^2\) in \(\mathbb{C}^n\). Let a positive constant \(\epsilon < 1\) be given. Then there exist positive constants \(C, \delta_0\) and \(C_s\) for all \(s \in (0, 1)\) such that, if \(u\) is a \(\delta\)-closed holomorphic \(p\)-form in the tube \(T_\delta = \mathbb{T}_\delta M\) for some \(0 < \delta \leq \delta_0\) and \(1 \leq p \leq n\), then:

(a) If \(p > m\), the equation \(dv = u\) has a holomorphic solution \(v\) in \(T_\delta\) satisfying

\[
\|v\|_{L^\infty(T_\delta)} \leq C\delta\|u\|_{L^\infty(T_\delta)}.
\]

(b) If \(p \leq m\) and the form \(i^*u\) is exact on \(M\), then for any solution of \(dv_0 = i^*u\) of class \(\Lambda^s(M)\) \((0 < s < 1)\) there is a holomorphic solution \(v\) of \(dv = u\) in \(T_\delta\) satisfying

\[
\|v\|_{L^\infty(T_\delta)} \leq C_s(\delta\|u\|_{L^\infty(T_\delta)} + \|v_0\|_{L^\infty(M)} + \delta^s\|v_0\|_{\Lambda^s(M)})
\]

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(c) If \( p \leq m \) and \( i^*u \) is exact on \( M \), there is a holomorphic solution of \( dv = u \) with
\[
\|v\|_{L^\infty(\mathcal{T}_\delta)} \leq C \|u\|_{L^\infty(\mathcal{T}_\delta)}.
\]

(5.3)

Remarks. 1. If \( \Omega \) is a Stein manifold, the Rham cohomology groups \( H^p(\Omega; \mathbb{C}) \) can be calculated by holomorphic forms in the following sense: Each closed form is cohomologous to a closed holomorphic \( p \)-form, and if a holomorphic form \( u \) is exact (i.e., \( u = dv_0 \) for some, not necessarily holomorphic, \( (p-1) \)-form \( v_0 \)), then also \( u = dv \) for a holomorphic \((p-1)\)-form \( v \) on \( \Omega \). (See [Hö], Theorem 2.7.10.)

2. On a \( C^2 \)-manifold \( M \), \( C^1 \)-forms and the \( d \)-operator \( d: C^0_p(M) \rightarrow C^0_{p-1}(M) \) are intrinsically defined. By duality, the notion \( dv = u \) (weakly) is well defined on \( M \). The condition in theorem 5.1 that \( i^*u \) be exact on \( M \) need only hold in the weak sense.

3. Theorem 5.1 has an extension to non-closed totally real \( C^1 \)-submanifolds \( M' \) in \( \mathbb{C}^n \). Let \( K \) be a compact subset of \( M' \) and let \( K' \subset M' \) be a compact neighborhood of \( K \) in \( M' \). (For instance, \( K = M \) may be a compact totally real submanifold with boundary in \( \mathbb{C}^n \).) For \( \delta > 0 \) we set
\[
U_\delta = \{ z \in \mathbb{C}^n : d_K(z) < \delta \}, \quad U'_\delta = \{ z \in \mathbb{C}^n : d_{K'}(z) < \delta \}.
\]

Choose \( c \in (0, 1) \). Assume that \( u \) is a \( d \)-closed holomorphic \( p \)-form in \( U'_\delta \), with \( i^*u \) exact on \( U'_\delta \cap M' \) (where \( i: M' \hookrightarrow \mathbb{C}^n \) is the inclusion map). Then there is a holomorphic solution of \( dv = u \) in \( U_{c\delta} \) such that the estimates (5.1), (5.2) and (5.3) are valid when \( \mathcal{T}_{c\delta} \) is replaced by \( U_{c\delta} \) and \( \mathcal{T}_\delta \) is replaced by \( U'_\delta \).

\[ \star \]

5.2 Lemma. Let \( 0 < c < c_1 < 1 \). Let \( u \) be a closed holomorphic \( p \)-form on \( \mathcal{T}_\delta \) for \( 0 < \delta \leq \delta_0 \) as in theorem 5.1. Suppose that there exists a differentiable \((p-1)\)-form \( v_1 \) on \( \mathcal{T}_{c_1\delta} \) satisfying \( dv_1 = u \) and
\[
\|v_1\|_{L^\infty(\mathcal{T}_{c_1\delta})} \leq A_\delta, \quad \|v_1\|_{L^\infty(\mathcal{T}_{c_1\delta})} \leq A_\delta \delta^{-s},
\]
where \( A_\delta \) depends on \( \delta \) and \( u \). Then there exists a holomorphic \((p-1)\)-form \( v \) in \( \mathcal{T}_{c\delta} \) satisfying \( dv = u \) and \( \|v\|_{L^\infty(\mathcal{T}_{c\delta})} \leq C_0 A_\delta \) for \( 0 < \delta \leq \delta_0 \), where \( C_0 \) is an absolute constant.
Proof of lemma 5.2. Let $v_1 = \sum_{q \leq q_0} v(q)$ where $v(q)$ is of bidegree $(p - 1, q, q)$. By comparing the terms of bidegree $(p - 1 - q_0, q_0 + 1)$ in the equation $dv_1 = \partial v_1 + \overline{\partial} v_1 = u$ and taking into account that $u$ is holomorphic, we see that $\overline{\partial} v(q_0) = 0$. If $q_0 > 0$, we get by theorem 3.1 a form $w$ on $\mathcal{T}_\delta$ solving $\overline{\partial} w = v(q_0)$ and satisfying the following estimates for some fixed $c < c_2 < 1$ and for all $1 \leq j \leq 2n$:

$$\|\partial_j w\|_{L^\infty(\mathcal{V}_j)} \leq C_1 \left( \|v(q_0)\|_{L^\infty(\mathcal{V}_j)} + \delta^s \|v(q_0)\|_{\Lambda^s(\mathcal{V}_j)} \right) \leq 2C_1A_\delta,$$

$$\|\partial_j w\|_{\Lambda^s(\mathcal{V}_j)} \leq C_1 \left( \|v(q_0)\|_{\Lambda^s(\mathcal{V}_j)} + \delta^{-s} \|v(q_0)\|_{L^\infty(\mathcal{V}_j)} \right) \leq 2C_1A_\delta \delta^{-s}.$$ 

Thus the form $v_2 = v_1 - w$ solves $dv_2 = dv_1 = u$, it only has components of bidegree $(p - q - 1, q)$ for $q < q_0$, and it satisfies

$$\|v_2\|_{L^\infty(\mathcal{V}_j)} \leq C' A_\delta, \quad \|v_2\|_{\Lambda^s(\mathcal{V}_j)} \leq C' A_\delta \delta^{-s}.$$ 

Repeated use of this argument gives a holomorphic solution of the equation $dv = u$ satisfying $\|v\|_{L^\infty(\mathcal{V}_j)} \leq C_0 A_\delta$. ♠

To prove theorem 5.1 it thus suffices to construct a good differentiable solution satisfying lemma 5.2, with $A_\delta$ as small as possible. Let $F = \{F_t\} : [0, 1] \times \mathcal{T}_0 \to \mathcal{T}_0$ be a $C^2$ deformation retraction of a tube $\mathcal{T}_0$ onto $M$, with $F_0$ the identity map and $\pi = F_0 : \mathcal{T}_0 \to M$ a retraction onto $M$. Let $i_\delta : M \to [0, 1] \times M$ be the map $x \to (t, x)$. By lemma 2.1 the form $\hat{v} = \int_0^1 i_* \left( \frac{\partial}{\partial t} \right) F^* u dt$ solves $\overline{\partial} \hat{v} = u - \pi^* u$ in $\mathcal{T}_\delta$. In the special local coordinates provided by lemma 2.4, if $u = \sum |I| + |J| = p u_I J dx^I \wedge dy^J$, the components of $\hat{v}$ are linear combinations of terms $y_j \int_0^1 i_* J^{-1} u_{I,J}(x, ty) dt$ for $j \in J$. Since the variables $y_j$ are transverse to $M$, we have $|y_j| = O(\delta)$ on $\mathcal{T}_\delta$ and hence $\|\hat{v}\|_{L^\infty(\mathcal{T}_\delta)} \leq C\delta \|u\|_{L^\infty(\mathcal{T}_\delta)}$, with $C$ independent of $\delta$. Replacing $\delta$ by $c_1 \delta$ and changing $C$ in each step below if necessary (but keeping it independent of $\delta$), it follows from Cauchy’s inequalities that $\|D\hat{v}\|_{L^\infty(\mathcal{V}_j)} \leq C_\delta^{-1} \|u\|_{L^\infty(\mathcal{T}_\delta)}$ and thus

$$\|D\hat{v}\|_{L^\infty(\mathcal{V}_j)} \leq C \|u\|_{L^\infty(\mathcal{T}_\delta)}, \quad \|\hat{v}\|_{\Lambda^s(\mathcal{V}_j)} \leq C \delta^{1-s} \|u\|_{L^\infty(\mathcal{T}_\delta)}.$$ 

In part (a) of theorem 5.1 we have $p > m$, and hence $\pi^* u = \pi^* (i^* u) = 0$ by degree reasons; so the form $v_1 = \hat{v}$ satisfies $dv_1 = u$ and the estimate (5.4) with $A_\delta = C\delta \|u\|_{L^\infty(\mathcal{T}_\delta)}$. Lemma 5.2 now completes the proof in this case.

To prove case (b) we set $v_1 = \hat{v} + \pi^* v_0$, where $v_0 \in \Lambda^s(M)$ solves $dv_0 = i^* u$. We get

$$\|v_1\|_{L^\infty(\mathcal{V}_j)} \leq C \left( \delta \|u\|_{L^\infty(\mathcal{T}_\delta)} + \|v_0\|_{L^\infty(M)} \right),$$

$$\|v_1\|_{\Lambda^s(\mathcal{V}_j)} \leq C \left( \delta^{1-s} \|u\|_{L^\infty(\mathcal{T}_\delta)} + \|v_0\|_{\Lambda^s(M)} \right).$$ 

Lemma 5.2 then provides a holomorphic solution of $dv = u$ satisfying the estimates (5.2).

Finally, to prove part (c) in theorem 5.1, we shall construct a good solution of $dv_0 = i^* u$ on $M$ belonging to $\Lambda^s(M)$ and apply (b). In order to circumvent problems caused by low differentiability of $M$ we use the following result of Whitney [Wh2]: If $M$ is a compact
$C^k$ manifold, $k \geq 1$, possibly with boundary, the underlying topological manifold may be given a structure of a $C^\infty$ manifold, denoted $M_0$, such that the set-theoretical identity map $i_0: M_0 \to M$ is a $C^k$-diffeomorphism.

Let $i_0: M_0 \to M$ be as above. We choose a smooth Riemann metric on $M_0$. We refer to Wells [We] for what follows. Let $d^*$ denote the Hilbert space adjoint of the exterior derivative $d$ with respect to the corresponding inner product on forms. The Laplace operator $\Delta = d^*d + dd^*$ has a corresponding Green’s operator $G: L^2_p(M_0) \to H^2_p(M_0))$ with the property that $\beta = d^*G(\alpha)$ is the solution of $d\beta = \alpha$ with minimal $L^2$-norm (orthogonal to the null-space of $\Delta$), provided that the equation $d\beta = \alpha$ is (weakly) solvable. For further details see section 4.5 in [We].

The Green’s operator is a classical pseudodifferential operator of order $-2$, so it induces bounded operators $L^\infty \to \Lambda^2$ and $\Lambda^s \to \Lambda^{s+2}$ for $s > 0$. (See [S], section VI, 5.3.) Now $i_0^*u$ is a $C^1$-form on $M_0$, and $v_0 = (i_0^{-1})^*(d^*Gi_0^*u)$ is a $C^1$-form on $M$ with $dv_0 = i^*u$, satisfying

$$\|v_0\|_{L^\infty(M)} \leq C_s\|u\|_{L^\infty(T_\delta)}.$$  

Substituting this into (5.2) gives (5.3).

**Remarks.**

1. In general the constant $C = C_\delta$ in the estimate (5.3) cannot be chosen so that $\lim_{\delta \to 0} C_\delta = 0$. To see this, let $i^*u \neq 0$ and choose a form $\phi \in C^1_{(m-p)}(M)$ with $\int_M u \wedge \phi \neq 0$. If $v_\delta$ solves $dv_\delta = u$ in $T_\delta$ and satisfies $\lim_{\delta \to 0} \|v_\delta\|_{L^\infty(T_\delta)} = 0$, we get

$$\int_M u \wedge \phi = \int_M dv_\delta \wedge \phi = \pm \int_M v_\delta \wedge d\phi \to 0$$

as $\delta \to 0$, a contradiction.

2. If $M$ is only of class $C^1$, the operator $d$ is not well defined on $M$. Instead, we call a $p$-form $\alpha$ on $M$ exact if there exists an integrable $(p-1)$-form $\beta$ on $M$ such that for each smooth $(m-p)$-form $\phi$ on a neighborhood of $M$ we have $\int_M \beta \wedge i^*(d\phi) = (-1)^p \int_M \alpha \wedge i^*\phi$. Then it is not hard to verify that $d(i_0^*\beta) = i_0^*\alpha$ (weakly) on $M_0$, and also $d(\pi^*\beta) = \pi^*\alpha$ on $T_\delta$. Using this, the proof carries over with only minor changes to the case when $M$ is of class $C^{1+\epsilon}$ for some $\epsilon > 0$, when $dv_0 = i^*u$ is interpreted as above.

3. If $M$ is of class $C^{2+\epsilon}$ for some $\epsilon > 0$, a more refined argument gives a holomorphic solution of $dv = u$ that also satisfies $|v|_{C^1(T_\delta)} \leq C \log(1/\delta)\|u\|_{L^\infty(T_\delta)}$ whenever $i^*u$ is exact. This reflects the fact that one expects to ‘gain almost a derivative’ in the interior estimates for the $d$-equation. We cannot establish such estimates with a constant independent of $\delta$. In fact, when $M = \{ z \in C^n : |z_j| = 1, 1 \leq j \leq n \}$, this would lead to the estimate $|\beta|_{C^1(M)} \leq const \|\alpha\|_{L^\infty(M)}$ for a solution of $d\beta = \alpha$, a contradiction.

4. Small changes are needed to prove theorem 5.1 when $M = K$ is a compact subset of a larger totally real $C^2$-submanifold $M' \subset C^n$ (see remark 3 following the statement of theorem 5.1). We follow the same proof as above, using the appropriate version of $\overline{\partial}$-results given by remark 3 following theorem 3.1. During the proof we shrink $K' \supset K$ and $\delta > 0$ several times. In the proof of (5.2), we observe that the $L^2$-minimal solution of $dv_0 = i^*u$ in $U_\delta \cap M'$ also satisfies $d^*v_0 = 0$ when $p > 1$, and we may apply the interior elliptic estimates.
to obtain Hölder estimates for $v_0$ in a neighborhood of $K$. There are also arguments to get the necessary control of $\|v_0\|_{L^2}$, for instance the Hodge decomposition in a manifold with boundary.

\&6. Proof of theorem 1.5.

In this section we prove theorem 1.5. We shall adapt a method of J. Moser [M] to the holomorphic setting.

Let $\omega$ be either the holomorphic volume form $dz_1 \wedge \cdots \wedge dz_n$ or the holomorphic symplectic form $\sum_{j=1}^n dz_2 dz_1 \wedge \cdots \wedge dz_n$, $n = 2n'$. Write $M = M_0$ and let $f: M = M_0 \rightarrow M_1$ be a $C^k$-diffeomorphism as in theorem 1.5 ($k \geq 2$), satisfying the condition (1.6) for some $C^{k-1}$-map $L: M \rightarrow GL(n, \mathbb{C})$. Let $i: M \hookrightarrow \mathbb{C}^n$ denote the inclusion. We assume in the proofs that $M$ is compact and without boundary. As usual we denote by $T_\delta = T_\delta M$ the tube of radius $\delta$ around $M$.

By lemma 2.6 there is a neighborhood $U \subset \mathbb{C}^n$ of $M$ and a $C^k$-diffeomorphism $\tilde{f}: U \rightarrow \tilde{f}(U) \subset \mathbb{C}^n$ extending $f$ such that $\tilde{f}$ is $\partial$-flat on $M$ and satisfies $(\tilde{f}^* \omega)_{z} = \omega_z$ at all points $z \in M$. The proof of theorem 1.2 then gives for each small $\delta > 0$ a holomorphic map $F_\delta': T_\delta \rightarrow \mathbb{C}^n$ of the form

$$F_\delta' = \tilde{f} + \delta R_\delta, \quad \|R_\delta\|_{C^j(T_\delta M)} = o(\delta^{k-j}); \quad 0 \leq j \leq k. \quad (6.1)$$

To prove theorem 1.5 we must construct biholomorphic maps $F_\delta$ as above which in addition satisfy $F_\delta^* \omega = \omega$. We need the following two lemmas.

6.1 Lemma. (Existence of a good $\partial$-flat extension.) If $\tilde{f}$ is any $\partial$-flat $C^k$-extension of $f$ satisfying $(\tilde{f}^* \omega)_{z} = \omega_z$ for all $z \in M$, there exists another $\partial$-flat $C^k$-extension $\hat{f}$ of $f$ satisfying $[\hat{f}^* \omega - \omega] = o(d^{k-1}_{M})$ near $M$ and $d\hat{f}_z = d\tilde{f}_z$ for all $z \in M$.

6.2 Lemma. (Approximation of a good $\partial$-flat extension.) Assume that $\tilde{f}$ is any $\partial$-flat $C^k$-extension of $f$ satisfying $[\tilde{f}^* \omega - \omega] = o(d^{k-1}_{M})$. Then for all sufficiently small $\delta > 0$ there exist biholomorphic maps $F_\delta: T_\delta \rightarrow \mathbb{C}^n$ with $F_\delta^* \omega = \omega$ and $\|F_\delta - \tilde{f}\|_{C^j(T_\delta M)} = o(\delta^{k-j})$ for $0 \leq j \leq k$.

We postpone the proof of lemmas 6.1 and 6.2 for a moment.

Proof of theorem 1.5 in the smooth case. Let $f: M = M_0 \rightarrow M_1$ be a $C^k$-diffeomorphism as in theorem 1.5. By lemma 2.6 there is a $\partial$-flat extension $\tilde{f}$ of $f$ satisfying $\tilde{f}^* \omega = \omega$ at points of $M$. By lemma 6.1 we can modify this extension, still denoting it $\tilde{f}$, such that $|\tilde{f}^* \omega - \omega| = o(d^{k-1}_{M})$. Finally we apply lemma 6.2 to get biholomorphic maps $F_\delta$ in tubes $T_\delta$ around $M$ satisfying $F_\delta^* \omega = \omega$ and the estimates (1.2). This proves theorem 1.5 in the smooth case, granted that lemmas 6.1 and 6.2 hold. We postpone the proof in the real-analytic case to the end of this section. ♠

Proof of Lemma 6.2. Let $\tilde{f}$ be as in lemma 6.2 and let $F_\delta': T_\delta \rightarrow \mathbb{C}^n$ (for small $\delta > 0$) be holomorphic maps of the form (6.1) obtained as in the proof of theorem 1.2. From the estimates on $R_\delta$ in (6.1) and the assumption $|\tilde{f}^* \omega - \omega| = o(d^{k-1}_{M})$ it follows that

$$\|(F_\delta')^* \omega - \omega\|_{C^j(T_\delta)} = o(\delta^{k-j-1}), \quad 0 \leq j \leq k - 1.$$
Set $\omega^\delta = (F^\delta_t)^* \omega$; this is a holomorphic $p$-form on $T_\delta$ which is close to $\omega$. Choose constants $0 < a < c < 1$. Using Moser's method [M] we shall construct a holomorphic map $G_\delta: T_{a\delta} \rightarrow T_\delta$ which is very close to the identity map and satisfies $G_\delta^* \omega^\delta = \omega$ on $T_{a\delta}$. The holomorphic map $F_\delta = F^\delta_t \circ G_\delta: T_{a\delta} \rightarrow \mathbb{C}^n$ is then close to $F^\delta_t$ (and hence to $\hat{f}$), and it satisfies $F_\delta^* \omega = G_\delta^* (\omega^\delta) = \omega$.

We first outline the Moser's method, postponing the estimates for a moment. Set $\omega^\delta = (F^\delta_t)^* \omega$ and $\omega^\delta_t = (1 - t) \omega + t \omega^\delta$ for $t \in [0, 1]$. Then $d\omega^\delta_t = 0$, and $\omega^\delta_t$ is close to $\omega$ for each $t$ and $\delta$. Our goal is to construct a $C^1$-family of holomorphic maps $G_t = G_{\delta,t}: T_{a\delta} \rightarrow T_\delta$ satisfying $G_0 = I$ and $G_t^* \omega^\delta_t = \omega$ for all $t \in [0, 1]$; the time-one map $G_\delta = G_{\delta,1}$ will then solve the problem.

To simplify the notation we suppress $\delta$ for the moment, writing $\omega^\delta_t = \omega_t$ and $G_{\delta,t} = G_t$. Suppose that such a flow $G_t$ exists. Denote by $Z_t$ its infinitesimal generator; this is a holomorphic time-dependent vector field on the image of $G_t$, satisfying $\frac{d}{dt} G_t(z) = Z_t(G_t(z))$ for each $t \in [0, 1]$ and each $z$ in the domain of $G_t$. Differentiating the equation $G_t^* \omega_t = \omega$ on $t$ and applying the time-dependent Lie derivative theorem ([AMR], Theorem 5.4.5., p. 372), we have

\[ 0 = \frac{d}{dt} (G_t^* \omega_t) = G_t^* (L_{Z_t} \omega_t + \frac{d}{dt} \omega_t) = G_t^* (d(Z_t) \omega_t) + \omega_1 - \omega. \] (6.2)

We have also used the Cartan’s formula for the Lie derivative $L_{Z_t} \omega_t$, as well as $d\omega_t = 0$. This shows that $G_t^* \omega_t = \omega$ holds for all $t \in [0, 1]$ if and only if the generator $Z_t$ satisfies the equation $d(Z_t) \omega_t + \omega_1 - \omega = 0$ for all $t \in [0, 1]$.

At this point we observe that $\omega$ is exact holomorphic on $\mathbb{C}^n$, $\omega = d\beta$; in fact when $\omega$ is the volume form (1.4) we may take $\beta = \sum_{j=1}^n (-1)^{j+1} d\bar{z}[j]$, and when $\omega$ is the symplectic form (1.5) we may take $\beta = \sum_{j=1}^n \bar{z}_{2j-1} d\bar{z}_{2j}$. Thus the equation $\omega_1 - \omega = F_\delta^* \beta - \beta = d(F_\delta^* \beta - \beta)$ is exact holomorphic on $T_\delta$. By theorem 5.1 we can solve the equation $dv = \omega_1 - \omega$ to get a small holomorphic $(p-1)$-form $v = v_1$ on $T_\delta$. Let $Z_t$ be the unique holomorphic vector field on $T_\delta$ solving the (algebraic!) equation $Z_t \omega_t + v = 0$. Integrating $Z_t$ we get a flow $G_t$ satisfying $G_t^* \omega_t = \omega$ on its domain of definition.

For this approach to work we must choose $\delta$ on $T_{a\delta}$ to have as small sup-norm as possible; this will imply that $|Z_t|$ is small, and hence its flow $G_t(z)$ will not escape the tube $T_{a\delta}$ (on which $Z_t$ is defined) before time $t = 1$, provided that the initial point $G_0(z) = z$ belongs to the smaller tube $T_{a\delta}$. (In particular, the solution $v_\delta = (F^\delta_t)^* \beta - \beta$ may not work since $F^\delta_t$ is not close to the identity map.)

In order to apply theorem 5.1 efficiently we must first show that $dv_0 = i^* (\omega_1 - \omega)$ has a solution on $M$ with small norm. Consider the map $h: [0, 1] \times M \rightarrow \mathbb{C}^n$, $h(t, z) = f(z) + tR_\delta(z)$, and set $w = h^* \omega$. Also let $i_t: M \rightarrow [0, 1] \times M$ denote the injection $i_t(z) = (t, z)$ ($z \in M$, $t \in [0, 1]$). It follows from lemma 2.1 that $v_0 = \int_0^1 i^*_t \left( \frac{d}{dt} [w] \right) dt$ solves $dv_0 = i^*_t w - i^*_0 w$. We have $i^*_t w = i^* \omega_1$ and $i^*_0 w = i^* \hat{f} \omega = i^* \omega$, so $dv_0 = i^* (\omega_1 - \omega)$. It follows from the formula above that $v_0 = \sum_{j=1}^n r_j^\delta v_j$, where $r_1^\delta, \ldots, r_n^\delta$ are the components of $R_\delta$ and $v_1, \ldots, v_n$ are $(p-1)$-forms on $M$ with $\|v_j\|_{C^k(M)}$ bounded independently of $\delta$. This gives $\|v_0\|_{C^l(M)} = o(\delta^{k-l})$ for $0 \leq l \leq k - 1$. It follows that $\|v_0\|_{L^\infty(M)} = o(\delta^{k-s})$ for
a given $s \in (0,1)$. Since $\|\omega_1 - \omega\|_{L^\infty(T_\delta)} = o(\delta^{k-1})$, it follows from theorem 5.1 that for all sufficiently small $\delta > 0$ we have a holomorphic solution of $dv_\delta = \omega_1 - \omega$ in $T_{c\delta}$, satisfying $\|v_\delta\|_{L^\infty(T_{c\delta})} = o(\delta^k)$.

Let $Z^\delta_t$ be the holomorphic vector field in $T_{c\delta}$ satisfying $Z^\delta_t|\omega_\delta = v_\delta$. The above estimate on $v_\delta$ implies $\|Z^\delta_t\|_{L^\infty(T_{c\delta})} = o(\delta^k)$, uniformly in $t \in [0,1]$. The standard formula for the rate of escape of the flow shows that we can choose $\delta_0 > 0$ sufficiently small such that for all $\delta \in (0, \delta_0)$ and all initial points $z \in T_{a\delta}$, the flow $G_{a\delta}(z)$ of $Z^\delta_t$ remains in $T_{c\delta}$ for all $t \in [0,1]$. At $t = 1$ we get a map $G_\delta = G_{a\delta,1}: T_{a\delta} \rightarrow T_{c\delta}$ satisfying $G_\delta^*\omega_1 = \omega$ and $|G_\delta(z) - z| = o(\delta^k)$ for $z \in T_{a\delta}$.

Set $F_\delta = G_\delta \circ F^\delta$. Since the maps $F^\delta$ have uniformly bounded $C^1$-norms on $T_{a\delta}$, we see that $\|F_\delta - F_\delta^\delta\|_{L^\infty(T_{a\delta})} = o(\delta^k)$. Replacing $a$ by a smaller constant and applying the Cauchy inequalities we also get

$$\|F_\delta - \bar{f}\|_{C^j(T_{a\delta})} \leq \|F_\delta - F^\delta\|_{C^j(T_{a\delta})} + \|F^\delta - \bar{f}\|_{C^j(T_{a\delta})} = o(\delta^{k-j}), \quad j \leq k.$$ 

By construction we have $F^*_\delta \omega = \omega$, so $F_\delta$ solves the problem.

\textbf{Remark.} This method applies on any domain $D \subset \subset C^n$ on which we can solve the $\overline{\partial}$-equations with estimates (e.g., on pseudoconvex domains); it shows that for any holomorphic map $F^\prime: D \rightarrow C^n$ for which $|F^\prime*\omega - \omega|$ is sufficiently uniformly small on $D$ there exists a holomorphic map $F: D' \rightarrow C^n$ on a slightly smaller domain $D' \subset \subset D$ such that $F^*\omega = \omega$ and $F$ is uniformly close to $F^\prime$ on $D'$. We obtain $F$ in the form $F = F^\prime \circ G$, where $G: D' \rightarrow D$ is a holomorphic map close to the identity, chosen such that $G^*(F^\prime*\omega) = \omega$.

The precise amount of shrinking of the domain depends on $\|F^\prime*\omega - \omega\|_{L^\infty(D)}$ and on the constants in the solutions of the $\overline{\partial}$-equations; we do not know if there is a solution to this problem on all of $D$.

We now turn to the proof of lemma 6.1. We shall need the following:

\textbf{6.3 Lemma.} Let $u$ be a $d$-closed $p$-form of class $C^{k-1}$ in a neighborhood of $M$, with $p \geq 1$, such that the $(p,0)$-component $u'$ of $u$ is $\overline{\partial}$-flat on $M$, and $u'' = u - u'$ is $(k-1)$-flat on $M$. Assume $i^*u = 0$, where $i: M \hookrightarrow C^n$ is the inclusion. Then there exists a $(p-1,0)$-form $v$ in a neighborhood of $M$ such that $v = \sum_{j=1}^N \zeta_j v_j$, where each $\zeta_j$ is a $\overline{\partial}$-flat $C^k$-function vanishing on $M$, each $v_j$ is a $\overline{\partial}$-flat $C^{k-1}$-form, and $|u - dv| = o(d_M^{k-1})$. If $u = 0$ on $M$, we may take $v_j = 0$ on $M$ for all $j$.

\textbf{Remark.} Using the rough multiplication (lemma 2.5) we see that there is a $\overline{\partial}$-flat $(p,0)$-form $v$ of class $C^k$ that also satisfies $|dv - u| = o(d_M^{k-1})$. However, the version stated above is often technically more convenient since we may wish to postpone the use of rough multiplication.

\textbf{Proof of Lemma 6.3.} In the case $m = n$ we may take $v = 0$ which can be seen as follows. We have $u' = \sum_{|I|=p} u_I dz^I$, where the coefficients $u_I$ are $C^k$-functions that are $\overline{\partial}$-flat on $M$; hence $i^*u = 0$ means that $u_I = 0$ on $M$ for all $I$ (since the coefficients of $u''$ vanish on $M$).
\( M \). It follows from the Cauchy-Riemann equations that each \( u_I \) is flat on \( M \), so we way choose \( v = 0 \).

When \( m < n \), we use the asymptotically holomorphic extension \( \widetilde{M} \) of \( M \) (lemma 2.4) and the \( \overline{\partial} \)-flat retraction \( F \) to \( \widetilde{M} \). Recall that a neighborhood of \( M \) may be covered by \( C^k \)-charts \( G_i : U_i \to V_i, G_i(z) = (z'_{(i)}(z), w'_{(i)}(z)) \in \mathbb{C}^m \times \mathbb{C}^{n-m}, 1 \leq i \leq r \), satisfying

- \( G_i \) is \( \overline{\partial} \)-flat on \( M \), \( G_i(M \cap U_i) = V_i \cap (\mathbb{R}^m \times \{0\}), G_i(\overline{M} \cap U_i) = V_i \cap (\mathbb{C}^m \times \{0\}) \);
- the retraction \( F \) is given in these local coordinates by \((t, (z', w')) \to (z', tw')\).

Let \( \tilde{i} : \widetilde{M} \to \mathbb{C}^m \) be the inclusion. Arguing as in the case \( m = n \) and making use of the \( \overline{\partial} \)-flat local parametrizations of \( \widetilde{M} \), we see that \( \tilde{i}^*u \) is flat on \( M \), and so is \( \tilde{\pi}^*u = \tilde{\pi}^*\tilde{i}^*u \), where \( \tilde{\pi} = F_0 \). When \( F : [0, 1] \times W \to W \) is the retraction to \( \widetilde{M} \), the form

\[
\hat{v} = \int_0^1 i_t^* \left( \frac{\partial}{\partial t} |F^*u| \right) dt
\]

(6.3) solves \( d\hat{v} = u - \tilde{\pi}^*u \) on a neighborhood of \( M \) according to lemma 2.1. Expressing \( u \) in the \( G_i \)-coordinates \((z'_{(i)}(z), w'_{(i)}(z))\) (which are \( \overline{\partial} \)-flat on \( M \)) we get

\[
u = \sum_{|I| + |J| = p} a_{I,J}(z'_{(i)}, w'_{(i)}) d\frac{z'^I}{z'_{(i)}} \wedge dw'^J_{(i)} + r'^I_{(i)}
\]

on \( U_i \), where the \( a_{I,J} \) are \( C^k \)-functions that are \( \overline{\partial} \)-flat on \( \mathbb{R}^m \times \{0\} \) and \( r'^I_{(i)} \) is a \( C^k \)-form that is flat on \( M \). Using the formula following lemma 2.1 we see that \( \hat{v} \) (6.3) is a linear combination of terms

\[
u_{(i),j} \left( \int_0^1 a_{I,J}(z'_{(i)}, tw'_{(i)}) |dK| dt \right) d\frac{z'^I}{z'_{(i)}} \wedge dw'^K_{(i)},
\]

where \( |I| + |K| = p - 1 \) and \( 1 \leq j \leq n - m \), plus a remainder term \( r''_{(i)} \) satisfying

\[
|\partial^\alpha r''_{(i)}| = o(d_M^{k-|\alpha|}) \text{ on } U_i \text{ for } |\alpha| \leq k - 1.
\]

Here \( u_{(i),j} \) denotes the \( j \)-th component of \( u'_{(i)} \).

Since \( G_{(i)} \) is \( \overline{\partial} \)-flat, it follows that

\[
\hat{v} = \sum_{j=1}^{n-m} \sum_{|I| = p-1} u'_{(i),j} g_{j,L}^{(i)} dz^L + r_{(i)}
\]

in \( U_i \), where each \( g_{j,L}^{(i)} \) is a \( \overline{\partial} \)-flat \( C^k \)-function and \( r_{(i)} \) behaves like \( r''_{(i)} \).

Choose a \( \overline{\partial} \)-flat partition of unity \( \{\psi_i\}_{i=1}^r \) subordinate to the covering \( \{U_i\}_{i=1}^r \), and choose \( \overline{\partial} \)-flat cut-off functions \( \chi_i \in C^\infty_0(U_i) \), with \( \chi_i = 1 \) near \( \text{supp } \psi_i \cap M \) for \( i = 1, \cdots, r \). Let \( \zeta_1, \cdots, \zeta_N \) (with \( N = r(n-m) \)) be some enumeration of the collection of functions \( \{\psi_i u'_{(i),j} : i \leq r, j \leq n - m\} \). Furthermore let \( v_1, \cdots, v_N \) be the corresponding enumeration of the forms \( \sum_{|I| = p-1} g_{j,L}^{(i)} dz^L \), prolonged by zero outside \( U_i \). Set \( v = \sum_{i=1}^N \zeta_i v_i \). Clearly
|dv - u| = o(d_M^{k-1}). Furthermore, if u = 0 on M, we also see that \( \int_0^1 a_{i,j}(z', tw') t |K| dt = 0 \) on \( V_t \cap (\mathbb{R}^m \times \{0\}) \), and hence \( v_1 = \cdots = v_N = 0 \) on M.

\[ \star \]

**Proof of Lemma 6.1.** In the unimodular case, \( \omega = dz_1 \wedge \cdots \wedge dz_n \), we could successively increase the order of vanishing of \( f \omega - \omega \) on M by adding certain correction terms to \( f \). This seems harder to do in the symplectic case, so we shall instead present an argument that works uniformly in both cases. It is a modification of Moser’s method: With \( \omega_t = (1 - t) \omega + tf^*\omega \), we shall construct a \( C^1 \)-family of \( \overline{\theta} \)-flat \( C^k \)-maps \( g_t \) on a neighborhood of \( M \), with \( g_0 = \text{id} \) and \( \frac{\partial}{\partial t} g_t^* \omega_t = o(d_M^{k-1}) \) uniformly in \( t \). Given such a family, integration in \( t \) gives \( \| g_t^* \omega_t - \omega \| = o(d_M^{k-1}) \). We will also show that \( g_1 \) is \( \overline{\theta} \)-flat on \( M \). Hence the map \( \hat{f} = f \circ g_1 \) will satisfy lemma 6.1. Furthermore, we shall see that \( |g_1(z) - z| = O(d_M(z)^2) \), so \( Dg_1 = Id \) on \( M \) and hence \( \hat{f} \) and \( f \) have the same differential on \( M \).

We shall obtain \( g_t \) by integrating a certain real time-dependent vector field \( X_t \) of class \( C^k \). Differentiating \( \frac{\partial}{\partial t} g_t^* \omega_t \) as in (6.2) we see that \( X_t \) must satisfy \( |d(X_t^* \omega_t) + \omega_1 - \omega| = o(d_M^{k-1}) \). We shall now construct such a vector field. More precisely, we shall construct a continuous family of \( C^k \) real vector fields \( X_t \) on a tube \( T_0 = T_{\delta_1} \), satisfying the following properties for each \( t \in [0, 1] \):

1. \( X_t \), considered as a map \( T_0 \to \mathbb{C}^n \), is \( \overline{\theta} \)-flat on \( M \). (Here we identify a real tangent vector \( X = \sum_{j=1}^n a_j \partial/\partial x_j + b_j \partial/\partial y_j \in T_z \mathbb{C}^n \) with the corresponding complex vector \( (a_1 + ib_1, \ldots, a_n + ib_n) \in \mathbb{C}^n \).)

2. \( |X_t(z)| \leq C d_M(z)^2 \) for some \( C > 0 \) independent of \( t \in [0, 1] \).

3. \( |d(X_t^* \omega_t) + \omega_1 - \omega| = o(d_M^{k-1}) \), uniformly in \( t \in [0, 1] \).

Let us first show that this solves the problem. We must show that \( X_t \) can be integrated from \( t = 0 \) to \( t = 1 \) for all initial values in a smaller tube. Recall that, after shrinking \( \delta_0 \) if necessary, the function \( d_M \) is differentiable in \( T_{\delta_1} \setminus M \), with a gradient of length one. Let \( z(t) \) be an integral curve of \( X_t \) in \( T_0 \setminus M \), \( t \in [0, t_0] \), and set \( u(t) = d_M(z(t)) \). Then

\[
u'(t) = \nabla d_M(z(t)) \cdot X_t(z(t)) \leq |X_t(z(t))| \leq C u(t)^2.
\]

Here we denoted by \( \nu \cdot \nu \) the real inner product of the vectors \( \nu, \nu \in \mathbb{C}^n \). Integrating the inequality \( u'(t)/u(t)^2 \leq C \) from 0 to \( t \) gives \( 1/u(0) - 1/u(t) \leq Ct \) and thus \( u(t)(1 - Ctu(0)) \leq u(0) \) for \( 0 \leq t \leq t_0 \). Let \( \delta_1 = \min(\delta_0/2, 1/2C) \). It follows that \( u(t) \leq u(0)/(1 - Ctu(0)) \leq 2u(0) \), and hence the integral curve extends to all values \( t \in [0, 1] \). Since \( |X_t(z(t))| \leq C u(t)^2 \), we see that \( |z(t) - z(0)| \leq 4C u(0)^2 t \). In other words, the time-t diffeomorphisms \( g_t \) are well defined on \( T_{\delta_1} \) for all \( t \in [0, 1] \) and they satisfy \( |g_t(z) - z| \leq 4C d_M(z)^2 \). In particular, \( g_t(z) = z \) and \( Dg_t(z) = Id \) for \( z \in M \) and \( t \in [0, 1] \).

To show that the \( C^k \)-maps \( g_t \) are \( \overline{\theta} \)-flat on \( M \), we consider the variational equation \( \frac{\partial}{\partial t} D_z g_t(z) = D_z X_t(g_t(z)) \circ D_z g_t(z) \) with the initial condition \( D_z g_0 = Id \). Decomposing the differential \( D\phi \) as the sum of a \( C \)-linear part \( D' \phi \) and a \( C \)-conjugate part \( D'' \phi \), we get

\[
\frac{\partial}{\partial t} D'_t g_t(z) = D'_t (\frac{\partial}{\partial t} g_t(z)) = D'_t (X_t(g_t(z)))
\]

\[
= (D'_t X_t)(g_t(z)) \circ D'_t g_t(z) + (D''_t X_t)(g_t(z)) \circ D'_t g_t(z).
\]
We apply both sides to a unit vector \( v \in \mathbb{C}^n \) and set \( y(t) = D_z'' g_t(z) v \in \mathbb{C}^n \). We obtain a linear differential equation \( y'(t) = A(t) y(t) + b(t) \) with the initial condition \( y(0) = D_z'' g_0(z) v = 0 \). The function \( u(t) = |y(t)| \) is differentiable when \( u(t) \neq 0 \) and \( u'(t) = y'(t) / |y(t)| \leq |y'(t)| \). Thus, if \( |A(t)| \leq A \) and \( |b(t)| \leq b \), we see that \( u'(t) \leq A u(t) + b \) where \( u(t) \neq 0 \). We shall prove that \( u(t) \leq b \left( e^{A t} - 1 \right) - \frac{b}{A} \) for \( t \in [0, 1] \). If \( u(t) = 0 \), there is nothing to prove. If not, let \( t_0 \) be the largest zero of \( u \) on the interval \([0, t]\). Then \( v(s) = u(s) a^{-A s} \) satisfies the differential inequality \( v'(s) \leq b e^{-A s} \) for \( s \in (t_0, t] \). Integration from \( t_0 \) to \( t \) gives \( v(t) \leq \frac{b}{A} \left( e^{-A t} - 1 \right) \) and \( u(t) \leq \frac{b}{A} \left( e^{A(t-t_0)} - 1 \right) \leq \frac{b}{A} \left( e^{A t} - 1 \right) \).

We know that \( |D_z X_t(z)| \) and \( |D_z g_t(z)| \) are bounded uniformly in \( z \in T_{\delta_t} \) and \( t \in [0, 1] \), while \( |D''_z X_t(z) = o(d_M(z)^{k-1}) \). Thus we may choose the upper bound \( A \) for \( |A(t)| \) independently of \( z \in T_{\delta_t} \) and \( t \in [0, 1] \), and the norm \( u \), we may choose the upper bound \( b \) of \( |b(t)| \) to be of size \( b = o(d_M(z)^{k-1}) \), uniformly in \( v \). Since \( u(t) = |D_z' G_t(z) v| \), it follows that \( |D''_z g_t(z)| = o(d_M(z)^{k-1}) \), so each \( g_t \) is \( \overline{\partial} \)-flat on \( M \).

By assumption we have \( |d(X_t) w_t| \geq (\omega_t - \omega_0) z = o(d_M(z)^{k-1}) \). Since \( d_M(g_t(z)) \leq 2d_M(z) \) and the norms \( |D_z g_t(z)| \) are bounded uniformly in \( z \in T_{\delta_t} \) and \( t \in [0, 1] \), we have \( |d''_z g_t(z)| = o(d_M(z)^{k-1}) \), uniformly in \( t \). By integration in \( t \) we obtain \( |g_t^\omega w_0 - \omega_0| = o(d_M(z)^{k-1}) \). Setting \( \hat{f} = f \circ g_t \), we see that \( \hat{f} \) is a \( \overline{\partial} \)-flat \( C^k \)-extension of \( f \), \( \hat{f} = Df \) on \( M \), and \( \left( \hat{f}^\omega - \omega \right) = o(d_M(z)^{k-1}) \). Thus \( \hat{f} \) satisfies lemma 6.1.

It remains to construct the vector field \( X_t \). Applying lemma 6.3 to \( \omega - \omega_1 \) we obtain a \( (p-1, 0) \)-form \( v \) near \( M \) with \( |dv - (\omega - \omega_1)| = o(d_M^{k-1}) \) and \( v = 0 \) on \( M \). We decompose \( \omega_t \) as \( \omega_t' + \omega_t'' \), where \( \omega_t' \) is the \( (p, 0) \)-component of \( \omega_t \). Then \( \omega_t' = \omega + t(\omega_t' - \omega) \), and \( \omega_t'' = \omega \) on \( M \) for each \( t \). Hence the map \( \phi : Z \rightarrow Z|\omega'_t \), taking the \( (1, 0) \)-vectors \( Z \in T_z^{1,0} \mathbb{C}^n \) to \( \Lambda^{p-1,0} T_z^* \mathbb{C}^n \), is an isomorphism for \( z \) near \( M \) and \( t \in [0, 1] \). Hence the equation \( Z'_t \|\omega_t = v \) uniquely defines a time-dependent \((1, 0)\) vector field \( Z'_t \) on \( \mathbb{C}^n \) near \( M \).

With respect to the basis \( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \) for \((1, 0)\)-vectors and the basis \( d z_1[1] \), \( \ldots, d z_n[n] \) (respectively \( d z_1, \ldots, d z_n \)) for \((p-1, 0)\)-covectors, the map \( \phi \) is represented by an \((n \times n)\)-matrix valued function \( A(t, z) = A_0 + t B(z) \), where \( A_0 \) is constant and invertible, and the entries of \( B(z) \) are \( \overline{\partial} \)-flat \( C^{k-1} \)-functions that vanish on \( M \). It follows that the entries of \( A(t, z)^{-1} \) are rational functions \( b(t, z) \) in \( t \), with coefficients that are \( \overline{\partial} \)-flat \( C^{k-1} \)-functions. From the properties of \( v \), as given by lemma 6.3, it follows that \( Z'_t = \sum_{j=1}^N \zeta_j \sum_{k=1}^n r_{jk}(t, z) \partial/\partial z_k \), where \( \zeta_1, \ldots, \zeta_N \) are \( C^k \)-functions that vanish on \( M \) and are \( \overline{\partial} \)-flat on \( M \), and each \( r_{jk} \) is a rational function in \( t \) with coefficients that are \( \overline{\partial} \)-flat \( C^{k-1} \)-functions with \( r_{jk}(t, z) = 0 \) for \( z \in M \).

We next apply the rough multiplication lemma to the pairs \( (\zeta_j(z), r_{jk}(t, z)) \) with respect to the compact subset \( M \times [0, 1] \) in \( \mathbb{C}^n \times \mathbb{R} \) and obtain \( C^k \)-functions \( a_l(t, z), 1 \leq l \leq n, \overline{\partial} \)-flat on \( M \) with respect to \( t \), such that \( |\sum_{j=1}^N \zeta_j r_{jk}(t, \cdot) - a_l(t, \cdot)| = o(d_M^{k-1}) \), uniformly in \( t \). (Remark: Use of the parametrized version of rough multiplication gives a smooth family of \( C^k \)-functions, but we do not need that.)

We set \( X_1 = \sum_{l=1}^n a_l(t, z) \partial/\partial z_l \) and \( X_2 = Z_2 + X_t \). Writing \( a_l = a_l + i v_l \), with \( u_l \) and \( v_l \) real, we have \( X_t = \sum_{l=1}^n u_l(z, t) \partial/\partial x_l + v_l(z, t) \partial/\partial y_l \). If we consider \( X_t \) as a map \( \mathbb{T}_0 \rightarrow \mathbb{C}^n \), this means that \( X_t = (a_1(t, z), \ldots, a_n(t, z)) \) and is \( \overline{\partial} \)-flat on \( M \). Furthermore, since \( \zeta_j(z) \) and \( r_{jk}(t, z) \) both vanish when \( z \in M \), we see that \( X_t \) vanish to the second order on \( M \).
Finally, we must show that (3) is satisfied. Writing \( X_t = Z_t + (Z_t - Z_t') + Z_t' \), we see that
\[
d(X_t|_{\omega_t}) + \omega_1 - \omega = d(Z_t'|_{\omega''_t}) + d((Z_t - Z_t')|_{\omega'_t}) + (dv + \omega_1 - \omega).
\]

The first term on the right hand side is \( o(d_M^k) \) since \( \omega''_t \) vanishes to order \( k-1 \) and \( Z_t \) vanishes to the second order on \( M \). Furthermore, \( Z_t - Z_t' \) vanishes to the \( k \)-th order, so the second term is \( o(d_M^{k-1}) \), and the third term is \( |dv + \omega_1 - \omega| = o(d_M^{k-1}) \). Thus (3) holds, uniformly in \( t \), since the derivatives are continuous in \((z,t)\).

\[\Box\]

**Proof of theorem 1.5 in the real-analytic case.** By assumption there is a continuous map \( \psi_0: M_0 \to SL(n, \mathbb{C}) \) (resp. \( \psi_0: M_0 \to SP(n, \mathbb{C}) \)) such that \( \psi_{0,z} \) agrees with \( dzf \) on \( T_zM_0 \) for each \( z \in M_0 \). By Remark 1 following the proof of theorem 1.2 (sect. 4) \( \psi_0 \) may be approximated by a holomorphic map \( \psi_1 \) from a neighborhood of \( M_0 \) to \( GL(n, \mathbb{C}) \) with \( \psi_{1,z} = dzf \) on \( T_zM_0 \) for each \( z \in \tilde{M}_0 \). Since \( \psi_{0,z}^*\omega = \omega \) for \( z \in M_0 \) and since \( \psi_1 \) approximates \( \psi_0 \) on \( M_0 \), it follows that the form \( \psi_{1,z}^*\omega = (\det \psi_{1,z})\omega \) is close to \( \omega \) for all \( z \in \tilde{M}_0 \) sufficiently near \( M_0 \).

We may think of \( \psi_1 \) as a holomorphic automorphism of the trivial bundle \( \tilde{M}_0 \times \mathbb{C}^n \to \tilde{M}_0 \). We claim that there is another holomorphic automorphism \( g \) of \( \tilde{M}_0 \times \mathbb{C}^n \) such that \( g|_{T\tilde{M}_0} = Id \) and \( g^*\psi_{1,z}^*\omega = \omega \). In the unimodular case we let \( g \) act as the identity on \( T\tilde{M}_0 \) and as multiplication by \( (\det \psi_1)^{-1/(n-m)} \) on \( \nu_0 \) (the holomorphic extension of the complex normal bundle \( \nu_0 \) to \( \tilde{M}_0 \)); the root is well-defined since the function \( \det \psi_{1,z} \) is close to \( 1 \). In the symplectic case \( g \) is a reduction to symplectic normal form with holomorphic dependence on \( z \in \tilde{M}_0 \). In both cases the map \( \psi = \psi_1 \circ g \) is an automorphism of the trivial bundle \( \tilde{M}_0 \times \mathbb{C}^n \) satisfying \( \psi^*\omega = \omega \).

Let \( F_1 \) be a biholomorphic extension of \( \tilde{f} \) constructed from \( \psi = \psi_1 \circ g \) as in Remark 1 (sect. 4), satisfying \( dzF_1 = \psi_z \) at points \( z \in M_0 \). Thus \( F_1^*\omega = \omega \) at points of \( M_0 \). Applying Moser’s method as above we can construct a biholomorphism \( G \) in a tubular neighborhood of \( \tilde{M}_0 \) which equals the identity on \( \tilde{M}_0 \) and satisfies \( G^*(F_1^*\omega) = \omega \). Then \( F = F_1 \circ G \) is a biholomorphic map near \( M_0 \) which extends \( f \) and satisfies \( F^*\omega = \omega \).

\[\Box\]

&7. Proof of theorems 1.7 and 1.8.

We will have to consider maps which have different degree of smoothness with respect to the time variable and the space variable, and we shall use the following terminology.

**Definition 4.** Let \( U \) be an open subset of \( [0,1] \times \mathbb{R}^m \). A mapping \( f: U \to \mathbb{R}^n \) is called a \( C^l \)-family of \( C^k \)-maps if \( \partial^j_x(\partial^k_yf) \) is continuous in \( U \) for \( 0 \leq j \leq l \) and \( |\alpha| \leq k \). There is an obvious extension of this notion to maps \( f: [0,1] \times M \to N \) where \( M \) and \( N \) are \( C^k \) manifolds. If in addition \( f_t = f(t \cdot) \) is a diffeomorphism (of its domain onto its image) for each \( t \in [0,1] \), we call \( f = \{ f_t \} \) a \( C^l \)-family of \( C^k \)-diffeomorphisms.

Thus a \( C^l \)-family of \( C^k \)-diffeomorphisms is the same as a \( C^k \)-isotopy (or a \( C^k \)-flow) in the sense of definition 1 in sect. 1. We remark that if \( f_t \) is a \( C^l \)-family of diffeomorphisms on domains \( U_t \subset \mathbb{R}^n \) for \( t \in [0,1] \), the family of inverses \( f_t^{-1} \) are not necessarily a \( C^l \)-family.
if \( l > 0 \), the reason being that the \( t \)-derivatives of the (derivatives of the) inverse map will involve higher order \( x \)-derivatives of the original map.

In the situation in Theorem 1.7 we shall say that a time-dependent family of \( C^k \)-forms on submanifolds \( M_t \subset \mathbb{C}^n \), \( \alpha_t = \sum_{I=0}^{n} \alpha_{I,t} dz^I \) with \( \alpha_{I,t} \in C^k(M_t) \), is a continuous family of \( C^k \)-forms if \( \alpha_{I,t} \circ f_t \) is a continuous family of \( C^k \)-functions on \( M \) for all multiindices \( I \). Recall that \( T_\delta = T_\delta M \) is the open tube of radius \( \delta \) around a submanifold \( M \subset \mathbb{C}^n \).

The main step in the proof of Theorem 1.7 is the following result.

7.1 Theorem. Let \( f_t : M = M_0 \to M_t \subset \mathbb{C}^n \) \((t \in [0,1])\) be a \( C^1 \)-family of \( C^k \)-diffeomorphisms between compact, totally real, \( C^k \)-submanifolds of \( \mathbb{C}^n \), with \( f_0 \) the identity on \( M \). By \( i_t : M_t \hookrightarrow \mathbb{C}^n \) we denote the inclusion map. Let \( \alpha_t \ (t \in [0,1]) \) be a continuous family of \((p,0)\)-forms of class \( C^k \) on \( M_t \) such that \( i_t^* \alpha_t \) is closed on \( M_t \) for each \( t \). Then there exists an extension of \( \alpha_t \) to a continuous family \( \hat{\alpha}_t \) of \((p,0)\)-forms of class \( C^k \) on a neighborhood of \( \hat{M} = \bigcup_{t \in [0,1]} \{ t \} \times M_t \) in \([0,1] \times \mathbb{C}^n \) such that for all sufficiently small \( \delta > 0 \) there exists a continuous family of closed holomorphic \( p \)-forms \( u_t^k \) on \( U_\delta = \bigcup_{t \in [0,1]} \{ t \} \times T_\delta M_t \) satisfying

\[
\| u_t^k - \hat{\alpha}_t \|_{C^r(T_\delta M_t)} = o(\delta^{k-r}), \quad 0 \leq r \leq k,
\]

uniformly in \( t \in [0,1] \). If \( i_t^* \alpha_t \) is exact on \( M_t \) for each \( t \in [0,1] \), we may choose \( u_t^k \) exact for every \( t \); in this case \( u_t^k \) can be chosen to be entire if each \( M_t \) is polynomially convex.

In the simplest case when \( M_t = M \) and \( \alpha_t = \alpha \) for all \( t \in [0,1] \), the main steps in the proof of theorem 7.1 are as follows (we write \( T_\delta = T_\delta M \)):

(i) We construct a \((p,0)\)-form \( \hat{\alpha} \) on a neighborhood of \( M \) such that \( d\hat{\alpha} \) is flat on \( M \). In particular, \( \hat{\alpha} \) is \( \overline{\partial} \)-flat on \( M \);

(ii) we approximate the coefficients of \( \hat{\alpha} \) by holomorphic functions to obtain a holomorphic \( p \)-form \( u' \) in \( T_\delta \) with \( \| du' \|_{L^\infty(T_\delta)} = o(\delta^{k-1}) \);

(iii) we solve \( dv = du' \), with \( v \) holomorphic and \( \| v \|_{L^\infty(T_\delta)} = o(\delta^k) \), and set \( u = u' - v \);

(iv) if \( i^* \alpha \) is exact, the norm of the de Rham cohomology class of \( i^* u \) is \( o(\delta^k) \), and this class may be represented by a holomorphic \( p \)-form \( u_0 \) on \( T_\delta \) of size \( o(\delta^k) \). Then \( u_1 = u - u_0 \) is exact and it approximates \( \alpha \) to the right order on \( M \).

In the parametric case we perform these steps such that the solutions are continuous with respect to the parameter \( t \). Before giving the proof of theorem 7.1 we summarize (slight extensions of) certain well-known results that we shall need.

We begin by considering the parameter dependence in Whitney’s extension theorem. Instead of a general compact subset \( K \subset \mathbb{R}^n \) (or \( K \subset \mathbb{C}^n \)) we consider the case when \( K \) is a compact \( C^1 \)-submanifold, with or without boundary. This is a so-called 1-regular set, so we have the following more precise results (see [T], chapter IV, sec. 1 and 2, in particular p. 76):

(i) Let \( A = \{ \alpha \in \mathbb{Z}^n_+ : |\alpha| \leq k \} \). The collections \( F = (f_\alpha)_{\alpha \in A} \subset C(K)^A \), satisfying the Whitney condition, form a closed subspace \( E^k(K) \) of \( C(K)^A \) with respect to the sup-norm; we shall call such collections Whitney functions.
(ii) The Whitney extension operator $W: \mathcal{C}^k(K) \to \mathcal{C}^k(K')$, where $K' \subset \mathbb{R}^n$ is a closed neighborhood of $K$, is linear and norm-continuous. Thus $\partial^\alpha W(F) = f_\alpha$ on $K$ for each $\alpha \in A$ and

$$\|W(F)\|_{\mathcal{C}^k(K')} \leq C \sup \{\|f_\alpha\|_{L^\infty(K)} : |\alpha| \leq k\}.$$ 

(iii) There exists a constant $C > 0$ such that $C \omega$ is a modulus of continuity for $\partial^\alpha W(F)$, $|\alpha| = k$, whenever $\omega$ is a modulus of continuity for all $f_\alpha$, $|\alpha| = k$.

From this it follows immediately that if $f_{\alpha,t}$, $\alpha \in A$, are $C^1$-families of continuous functions on $K$ and if $F_t = (f_{\alpha,t})_{\alpha \in A}$ is a Whitney function for each $t \in [0, 1]$, then their Whitney extensions $W(F_t)$ are a $C^1$-family of $C^k$-functions, and we may bound the $t$ and $x$ derivatives of $W(F_t)$ in terms of $F_t$.

Using the above results, the proof of lemma 2.5 (sec. 2) gives the following:

**7.2 Lemma.** (Parameter-dependent rough multiplication.) Let $K \subset \mathbb{R}^n$ be a compact $C^1$-submanifold, with or without boundary. Let $f_t$ be a $C^1$-family of $C^k$-functions and $g_t$ a $C^1$-family of $C^{k-1}$-functions on a neighborhood of $K$ in $\mathbb{R}^n$ such that $f_t = 0$ on $K$ for each $t \in [0, 1]$. Then there exists a $C^1$-family of $C^k$-functions $h_t$ on a neighborhood of $K$ such that $|h_t - f_t g_t| = o(d^k)$, uniformly in $t \in [0, 1]$. If $K \subset \mathbb{C}^m$ and if $f_t, g_t$ are $\overline{\partial}$-flat on $K$, then so is $h_t$.

We next prove an extension lemma.

**7.3 Lemma.** Let $M \subset \mathbb{C}^n$ be a compact, totally real, $C^k$-submanifold. For any $C^1$-family of $C^k$-maps $f_t: M \to \mathbb{C}^N$ ($t \in [0, 1]$) there exist an open set $U \subset \mathbb{C}^n$ containing $M$ and a $C^1$-family of $C^k$-maps $\tilde{f}_t: U \to \mathbb{C}^N$ such that each $\tilde{f}_t$ is $\overline{\partial}$-flat on $M$ and it restricts to $f_t$ on $M$. If $N = n$ and $f_t: M \to M_t = f_t(M) \subset \mathbb{C}^n$ is a diffeomorphism for each $t \in [0, 1]$, we can choose $f_t$ as above to be a $C^1$-family of $C^k$-diffeomorphisms on $U$.

**Proof of Lemma 7.3.** Let $m = \dim \mathbb{R} M \leq n$. We consider first the case when $M = \overline{V}$ is a smoothly bounded compact domain in $\mathbb{R}^m \subset \mathbb{C}^m \subset \mathbb{C}^n$. Write $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$. Given $f \in C^k(\overline{V})$, we consider the following Whitney function on $\overline{V}$ for the real coordinates $x_1, \ldots, x_m, y_1, \ldots, y_m$ in $\mathbb{C}^m$:

$$F: f(a', a'') = i|\alpha'\alpha''| \partial_{x_j}^{\alpha'} \partial_{y_j}^{\alpha''}(f), \quad \alpha', \alpha'' \in \mathbb{Z}_+^m, |\alpha'| + |\alpha''| \leq k.$$ 

From the Cauchy-Riemann equations $\frac{\partial g}{\partial y_j} = i \frac{\partial g}{\partial x_j}$ (1 $\leq j \leq m$) for a function $g$ in a neighborhood of $\overline{V}$ in $\mathbb{C}^m$ it follows that the Whitney extension $\tilde{f} = W(F)$ of $F$ to $\mathbb{C}^m$ is $\overline{\partial}$-flat on $\overline{V}$. If $m < n$, we extend $W(F)$ trivially in the variables $z_{m+1}, \ldots, z_n$ to get a Whitney extension on $\mathbb{C}^n$. Moreover, if $\{f_t: t \in [0, 1]\}$ is a $C^1$-family of $C^k$-functions on $\overline{V}$ and we define $F_t$ as above, the Whitney extensions $W(F_t)$ are a $C^1$-family of $C^k$-functions which are $\overline{\partial}$-flat on $\overline{V}$.

Next we consider a local $C^k$-parametrization $\phi: U \to M$ around a point $w_0 \in M$, where $U$ is an open set in $\mathbb{R}^m$. Let $z_0 = \phi^{-1}(w_0) \in U$. Choose a smoothly bounded
domain $V \subset \subset U$ containing $z_0$ and set $W = \phi(V) \subset M$. Let $\tilde{\phi}$ be an extension of $\phi$ to $\mathbb{C}^n$ constructed above which is $\overline{\partial}$-flat on $\overline{V}$. If $m < n$, we also choose a basis $v_1, \ldots, v_{n-m}$ of the complex normal space $(T^C_{w_0}M)_\perp$ to $M$ at $w_0$. The map $\Phi(z) = \tilde{\phi}(z) + \sum_{j=1}^{n-m} z_{m+j}v_j$ is then a $C^k$-diffeomorphism in a neighborhood of $z_0$ which is $\overline{\partial}$-flat on $\overline{V}$; hence its inverse $\Phi^{-1}$ is well defined in a neighborhood $\overline{W} \subset \mathbb{C}^n$ of $w_0$ and is $\overline{\partial}$-flat on $W \cap \overline{W} \subset M$.

The first part of the proof also provides an extension $\psi_t$ of the map $f_t \circ \phi: \overline{V} \to \mathbb{C}$ to a neighborhood of $\overline{V}$ in $\mathbb{C}^n$ such that $\psi_t$ is $\overline{\partial}$-flat on $\overline{V}$. The composition $\psi_t \circ \Phi^{-1}: \overline{W} \to \mathbb{C}$ is a $C^k$-extension of the map $f_t$ which is $\overline{\partial}$-flat on $W \cap \overline{W} \subset M$.

This gives us a local $\overline{\partial}$-flat $C^k$-extension of $f_t$ in a neighborhood of each point $w_0 \in M$. We can patch these local extensions by a $\overline{\partial}$-flat partition of unity along $M$ as in lemma 2.6 to obtain a desired $C^l$-family $\hat{f}_t$ satisfying lemma 7.3.

It remains to consider the case when $f_t: M \to M_t$ is a diffeomorphism for each $t \in [0, 1]$. Let $\hat{M} = \bigcup_{t \in [0, 1]} \{t\} \times M_t \subset [0, 1] \times \mathbb{C}^n$, and let $\hat{f}: [0, 1] \times M \to \hat{M}$ be the map $f(t, z) \to (t, f_t(z))$. Let $\nu$ denote the complex normal bundle of $M$ and $\nu^t$ the complex normal bundle of $M_t$ in $\mathbb{C}^n$. Then $\hat{\nu} = \bigcup_{t \in [0, 1]} \{t\} \times \nu^t$ is, in an obvious way, a vector bundle over $\hat{M}$, and $\nu \times [0, 1]$ is a vector bundle over $[0, 1] \times M$. By standard bundle theory (see Lemma 1.4.5 of [Ati]) there exists a bundle equivalence $\psi: [0, 1] \times \nu \to \hat{\nu}$ over $\hat{f}$. Thus we have continuously varying isomorphisms $\nu_z \to \overline{\nu}^{f_t(z)}$ $(z \in M, t \in [0, 1])$ which we extend to a continuous map $A': [0, 1] \times M \to \text{End}_C(\mathbb{C}^n)$. Then we approximate $A'$ by a $C^l$-family of $C^k$-maps $A: [0, 1] \times M \to \text{End}_C(\mathbb{C}^n)$ so that $A(t, z)\nu_z$ is a supplementary subspace to $(Df_t)_z(T^C_zM)$ for each $(t, z) \in [0, 1] \times M$. Let $L(t, z)$ be $(Df_t)_z^{C}$ on $T^C_zM$ and $A(t, z)$ on $\nu_z$. Since $T_z\mathbb{C}^n = T^C_zM \oplus \nu_z$, $L(t, z)$ belongs to $GL(n, \mathbb{C})$, and it is not hard to check that $L_t = L(t, z): M \to GL(n, \mathbb{C})$ is a $C^l$-family of $C^{k-1}$-maps extending $Df_t$. Using Lemma 7.2 it is easy to see that lemma 2.6 has a parameter-dependent version which gives the desired conclusion.

\*\* Proof of Theorem 7.1. \*\*

Set $M = M_0$ and $i = i_0: M \hookrightarrow \mathbb{C}^n$. We first apply Lemma 7.3 to get a neighborhood $U \subset \mathbb{C}^n$ of $M$ and a continuous family of $C^k$-diffeomorphisms $\hat{f}_t: U \to U_t \subset \mathbb{C}^n$ which are $\overline{\partial}$-flat on $M$. The family of inverses $(\hat{f}_t)^{-1}: U_t \to U$ is then a continuous family of $C^k$-diffeomorphisms on $U = \bigcup_{t \in [0, 1]} \{t\} \times U_t$ which are $\overline{\partial}$-flat on $M_t$ and which extend $\overline{\partial}^{-1}: M_t \to M$.

Let $\alpha_t = \sum_{|I|=p} \alpha_{I,t}dz^I$ be as in theorem 7.1, with $\alpha_{I,t} \in C^k(M_t)$. Our assumption is that $\alpha_{I,t} \circ \hat{f}_t$ is a $C^k$-continuous family of $C^k$-functions for each $I$. Applying lemma 7.3 we can extend it to a continuous family $\alpha'_{I,t}$ of $C^k$-functions on $[0, 1] \times U$ which are $\overline{\partial}$-flat on $M$. Set $\alpha_{I,t} = \alpha'_{I,t} \circ (\hat{f}_t)^{-1}$ and $\alpha_t = \sum_{|I|=p} \alpha_{I,t}dz^I$; this is a continuous family of $C^k$ $(p, 0)$-forms on $U$, and $\alpha_t$ is $\overline{\partial}$-flat on $M_t$.

The next step is to modify $\alpha_t$ so as to make its differential flat on $M_t$. We observe that both $\alpha_t^* \alpha_t$ and $\beta_t = d\alpha_t^* \alpha_t = \hat{f}_t^* (d\alpha_t)$ are continuous families of $C^{k-1}$-forms on $U$. By assumption, $d\alpha_t^* \alpha_t = 0$, hence $\beta_t = 0$.

It is clear that the proof of lemma 6.3 produces a $C^l$-family of solutions $\nu_t$ for any $C^l$-family $u_t$ satisfying the assumptions in that lemma. Applying this to the forms $u_t = \beta_t$
constructed above, we obtain a continuous family of \((p,0)\)-forms \(\gamma_t^j = \sum_{j=1}^{N} \zeta_j \gamma_t^{j,t} (t \in [0,1])\) such that \(d\gamma_t^j - \beta_t\) is \((k - 1)\)-flat on \(M\), where \(\zeta_1, \ldots, \zeta_N\) are \(\overline{\partial}\)-flat \(C^k\)-functions vanishing on \(M\) and \(\gamma_t^{j,t}\) are continuous families of \(C^{k-1}\) \((p,0)\)-forms that are \(\overline{\partial}\)-flat on \(M\).

Then \((\tilde{f}_t^{-1})^*\gamma_t^{j,t} = \sum_{|\lambda| = p} a_{j,t} \lambda dz^j + \lambda_t dt\) where \(a_{j,t}\) are continuous families of \(C^{k-1}\)-functions that are \(\overline{\partial}\)-flat on \(M_t\) and \(\lambda_t = o(d\delta_{M_t})\) uniformly in \(t\). Applying parameter dependent rough multiplication (lemma 7.2) to \(\zeta_j\) and \(a_{j,t} \circ \tilde{f}_t\) gives continuous families \(b_{j,t}\) of \(C^k\)-functions near \(M_t\) which are \(\overline{\partial}\)-flat on \(M_t\). Setting \(\gamma_t = \sum_{|\lambda| = p}^{N} (b_{j,t} \circ \tilde{f}_t^{-1}) dt\) and \(\tilde{\alpha}_t = \tilde{\alpha}_t - \gamma_t\), we get \(\tilde{\alpha}_t |_{M_t} = \alpha_t, t \in [0,1]\), and \(d\tilde{\alpha}_t = o(d\delta_{M_t})\) uniformly in \(t\).

The next step is to approximate \(\tilde{f}_t\) well by biholomorphic maps in tube neighborhoods \(T_\delta\) of \(M\). \(\hat{f}_t\) maps \(M_t\) onto \(M_t\) and is a diffeomorphism from a neighborhood \(U_t\) of \(M_t\) with estimates on derivatives valid for all \(t \in [0,1]\). It follows that for some \(\pi > 0\) and all sufficiently small \(\delta > 0\) we have \(\hat{f}_t(T(\pi T_\delta)M_t) \subset T_\delta M_t\) and \(\hat{f}_t^{-1}(T(\pi T_\delta)M_t) \subset T_\delta M_t\) for all \(t \in [0,1]\).

If we apply the solution operator of theorem 3.1 to the equation \(\overline{\partial}R_\delta^t = \overline{\partial}f \tilde{f}_t\) in \(T_\delta = T_\delta M\) and set \(h_\delta^t = \tilde{f}_t - R_\delta^t\), we obtain a continuous family of holomorphic maps \(h_\delta^t\) on \(T_\delta\) satisfying \(\|h_\delta^t - \tilde{f}_t\|_{C^k(T_\delta M)} = o(\delta^{k-j})\) for \(j \leq k\), where \(k \geq 2\). It follows that for small \(\delta > 0\) the map \(h_\delta^t\) is a biholomorphism of \(T_\delta\) onto its image, and \(g_t^\delta := \tilde{f}_t^{-1} \circ h_\delta^t\) is a \(C^k\)-diffeomorphism of the tube \(T_\delta\) onto a small perturbation of \(T_\delta\).

Since \(h_\delta^t\) is close to \(\tilde{f}_t\), it is not hard to see, using the argument in the proof of theorem 1.2, that if \(0 < a < \pi\) and \(\epsilon > 0\) are given then for \(\delta > 0\) sufficiently small (depending on \(a\) and \(\epsilon\)) we have the inclusions \(h_\delta^t(T(\pi T_\delta)M_t) \subset T_\delta M_t\) for \(\epsilon \delta \leq \delta' \leq \delta\) and \((h_\delta^t)^{-1}(T_\delta M_t) \subset T_\delta M_t\) for \(\epsilon \delta \leq \delta' \leq a \delta\). We also have \(T_\delta M_t \subset T_{\delta/2} M_t \subset T_{\delta a} M_t\) for \(\epsilon \delta \leq \delta' \leq \delta\), for all \(t\).

The next step is to approximate \(\tilde{\alpha}_t\) by a continuous family of holomorphic \(p\)-forms \(u_t^\delta = u_t^\delta\) on tubes \(T_\delta M_t\). Suppose that \(\tilde{\alpha}_t = \sum_{|\lambda| = p} \tilde{\alpha}_t, t \in [0,1]\). For small \(\delta > 0\), \(h_\delta^t/a(T(\pi T_\delta)M_t) \subset T_\delta M_t\) for \(t \in [0,1]\). Let \(u_t^\delta, t\) be holomorphic approximations to \(\tilde{\alpha}_t, t \circ h_\delta^t/a\), constructed as \(T_\delta\) in section 4. Set \(u_t^\delta, t = u_t^\delta, t \circ h_\delta^t/a\). Then the \(p\)-form \(u_t^\delta = \sum_{|\lambda| = p} u_t^\delta, t dz^j\) is holomorphic in \(T_\delta M_t\) and satisfies \(\|u_t^\delta - \tilde{\alpha}_t\|_{C^k(T_\delta M_t)} = o(\delta^{k-j})\), uniformly in \(t\). We also see that \(\|du_t^\delta\|_{L^\infty(T_\delta M_t')} = o(\delta^{k-1})\), and if we set \(v_{0,t} = i^* u_t^\delta - \tilde{\alpha}_t\) then \(dv_{0,t} = i^* du_t^\delta\).

We wish to prove the existence of a continuous family of holomorphic \((p - 1)\)-forms \(v_t = v_t^\delta\) on \(T_\delta M_t\) for some \(b > 0\), with \(\|v_t\|_{L^\infty(T_\delta M_t')} = o(\delta^{k})\), uniformly in \(t\), and solving \(dv_t = du_t^\delta\). Then \(u_t^\delta = u_t^\delta - v_t\) would be a continuous family of closed holomorphic \(p\)-forms with \(\|u_t|_{M_t} - \alpha_t\|_{C^k(M_t')} = o(\delta^{k-j})\), uniformly in \(t\), as required.

A parameter-dependent version of theorem 5.1 for the family \(M_t\) would give that result. The following argument will give this for a \(b > 0\), but we shall restrict ourselves to the special case we need. Choose \(a < \pi\) and \(\epsilon = a/2\). For \(\delta > 0\) small, \(u_t^\delta = i^* du_t^\delta\) are \(C^k\)-forms on \(T_\delta M\) with \(\|u_t^\delta\|_{L^\infty(T_\delta M)} = o(\delta^{k-1})\) and \(\|u_t^\delta\|_{C^k(T_\delta M)} = o(\delta^{k-1})\), uniformly in \(t\).

Furthermore, with \(v_{0,t}^\delta = f_t^* v_{0,t}\), we have \(dv_{0,t}^\delta = i^* u_t^\delta\) on \(M\), with \(\|v_{0,t}^\delta\|_{L^\infty} = o(\delta^{k})\)
and $\|u_0^t\|_{C^s} = o(\delta^{k-s})$, uniformly in $t$. Then the first part of the proof of theorem 5.1 and the remarks on continuous $t$-dependence give a continuous family of $C^{k-1}$-forms $\omega'_t$ on $T_{a\delta}M$ solving $d\omega'_t = u'_t$, with $\|\omega'_t\|_{L^\infty} = o(\delta^k)$ and $\|\omega'_t\|_{C^s} = o(\delta^{k-s})$, uniformly in $t$. Then $\omega_t = (g_t^s)^*\omega'_t$ are defined on $T_{a\delta/2}M$ and satisfy the same kind of estimates, and $d\omega_t = (h_t^s)^*(\tilde{f}_t^{-1})^*u'_t = (h_t^s)^*d\omega'_t$ is holomorphic. Since $a < \bar{a}$, the second part of the proof of theorem 5.1 gives the existence of a continuous family of holomorphic $p$-forms $\nu_t^s$ on $T_{a\delta/2}M$ satisfying $d\nu_t^s = (h_t^s)^*d\omega'_t$ and $\|\nu_t^s\|_{L^\infty(T_{a\delta/2}M)} = o(\delta^k)$, uniformly in $t$. By assumption $h_t^s(T_{a\delta/2}M) \supset T_{a\delta/2}M_t$ for each $t$, and $\nu_t^s = (h_t^s)^{-1}*u'_t$ is a continuous family of holomorphic $p$-forms on $T_{a\delta/2}M_t$ with $d\nu_t^s = d\omega'_t$ on $T_{a\delta/2}M_t$ and $\|\nu_t^s\|_{L^\infty} = o(\delta^k)$ uniformly in $t$.

We now show that if $i_t^*\alpha$ is exact for every $t$, the holomorphic forms $u_t^s$ as above may be chosen to be exact. We recall that the de Rham cohomology group $Hp(M, C)$ is finite dimensional and $f_t^*: HP(M_t, C) \to HP(M, C)$ is an isomorphism for every $t$. We have that $HP(M, C) \cong \{\alpha \in C_0(M) : d\alpha = 0\}/(\text{exact forms})$, where derivatives are taken in the weak sense, and we may equip $HP(M, C)$ with the quotient norm.

For each $t_0 \in [0, 1]$ there exist closed holomorphic $p$-forms $\hat{\omega}_1, \ldots, \hat{\omega}_N$ on an open neighborhood $U$ of $M_{t_0}$ such that $[i_{t_0}^*\hat{\omega}_j], 1 \leq j \leq N$, is a basis for $HP(M_{t_0}, C)$. Then $t \to [f_t^*\hat{\omega}_j]$ is a continuous map $[0, 1] \to HP(M, C)$, and $t \to [\hat{f}_t^*\hat{\omega}_j]$ is continuous for $t$ near $t_0$ and $1 \leq j \leq N$. It follows that $[[f_t^*\hat{\omega}_j]: j \leq N]$ is a basis for $HP(M, C)$ for $t$ in a neighborhood $J \subset [0, 1]$ of $t_0$, and that we may write $[f_t^*\hat{\omega}_j] = \sum_{j=1}^N c_j^t(t) [\hat{f}_t^*\hat{\omega}_j]$ with $c_j^t$ continuous on $J$. Each form $\hat{f}_t^*\alpha_t$ is exact on $M$, so $\|[[f_t^*\hat{\omega}_j]]\|_{L^\infty(M)} = o(\delta^k)$. This means that for $J_1 \subset J$ we have $max_{t \in J_1} |c_j^t(t)| = o(\delta^k)$ for all $j \leq N$. For $\delta > 0$ small and $t \in J_1$ we have $T_{a\delta}M_t \subset U$, $u_{t^0}\delta^s = u_{t^0}^s - \sum_{j=1}^N c_j^t(t) \hat{\omega}_j$ is exact on $T_{a\delta}M_t$ (since $[i_{t_0}^*\hat{\omega}_j] = 0$), and it approximates $\alpha_t$ well enough. We can now patch these together with a part of unity in $t$ to obtain a solution $u_t^s$ for $t \in [0, 1]$ satisfying theorem 7.1.

Finally, assume $M_t$ is polynomially convex for all $t \in [0, 1]$ and let $u_t^s$ be the exact solution on $U_\delta$. For $\delta > 0$ sufficiently small we may also assume that $T_{a\delta}M_t$ is Runge in $C^\infty$ for all $t$. Given $a < d < 1$ and $\epsilon > 0$, there exist $t_j \in [0, 1], j = 1, \ldots, N$, and (relatively) open intervals $I_j \subset [0, 1], j \leq I_j$, such that $U_{a\delta} \subset \bigcup_{j=1}^N I_j \times T_{a\delta}M_{t_j} \subset U_\delta$, and for all $t \in I_j$ we have $\|u_t^s - u_{t_j}^s\|_{C^s(T_{a\delta}M_{t_j})} < \epsilon$ and $\|\omega_t^s - \omega_{t_j}^s\|_{C^s(T_{a\delta}M_{t_j})} < \epsilon$. Let $\beta_j$ be a holomorphic $(p-1)$-form on $T_{a\delta}M_t$ such that $d\beta_j = u_{t_j}^s$. By Oka's theorem there is an entire $(p-1)$-form $v_j$ such that $\|\beta_j - v_j\|_{L^\infty(T_{a\delta}M_{t_j})} < \epsilon$. The Cauchy estimates imply $\|\beta_j - v_j\|_{C^s(T_{a\delta}M_{t_j})} = o(\delta^{-r})$, and hence $\|u_{t_j}^s - d\omega_{t_j}^s\|_{C^{r}(T_{a\delta}M_{t_j})} = o(\delta^{-r+1})$. Choosing $r = o(\delta^{-k+1})$, we obtain $\|u_{t_j}^s - \omega_{t_j}^s\|_{C^{r}(T_{a\delta}M_{t_j})} = o(\delta^{-k-r})$ whenever $t \in I_j$. If $x_j(t)$ is a partition of unity on $[0, 1]$ subordinate to the covering $\{I_j\}$ and we define $v_t = \sum_{j=1}^N x_j(t)v_j(z)$, then $u_t = dv_t$ is an entire form for each $t$ which satisfies theorem 7.1.

\textbf{Proof of Theorem 1.7.} By assumption $f_t: M \to M_t$ is $C^1$-family of $C^k$-diffeomorphisms and $\omega$ is one of the forms (1.4), (1.5). Let $X_t$ be the infinitesimal generator of $f_t$, i.e., $\partial_t f_t(z) = X_t(f_t(z))$ for $z \in M$ and $t \in [0, 1]$. Then $\alpha_t = X_t|\omega$ is a continuous family of $(p, 0)$-forms on $M_t$, with $p = n-1$ when $\omega$ is the volume form (1.4) and $p = 1$ when $\omega$
is the symplectic form \((1.5)\). Since \(f_t\) is an \(\omega\)-flow, \(i_{f_t}^*\alpha_t\) is closed on \(M_t\) for each \(t\), by the remark after definition 2.

By theorem 7.1 there exists an extension of \(\alpha_t\) to a continuous family \(\tilde{\alpha}_t\) of \((p,0)\)-forms of class \(C^k\) on a neighborhood of \(\tilde{M} = \bigcup_{t \in [0,1]} \{t\} \times M_t\) such that for all sufficiently small \(\delta > 0\) there exists a continuous family of closed holomorphic \(p\)-forms \(u_t^\delta\) on \(U_\delta = \bigcup_{t \in [0,1]} \{t\} \times T_\delta M_t\) with \(\|u_t^\delta - \tilde{\alpha}_t\|_{C^r(T_0 M_t)} = o(\delta^{k-r})\), uniformly in \(t\), for \(0 \leq r \leq k\).

The equation \(u_t^\delta = Y_t^\delta \omega\) uniquely defines a time-dependent holomorphic vector field \(Y_t^\delta\) on \(U_\delta\). Since \(u_t^\delta\) is closed, the flow \(F_t^\delta\) of \(Y_t^\delta\) is a holomorphic \(\omega\)-flow wherever it is defined (see definition 2). If we let \(X_t\) denote the extension of \(X_t\) to \(U_\delta\) defined by \(\tilde{\alpha}_t = X_t\omega\), then \(\|Y_t^\delta - X_t\|_{C^r(T_0 M_t)} = o(\delta^{k-r})\), uniformly in \(t\). We may apply lemma 4.1 of [FL] to see that for small \(\delta > 0\) the flow \(F_t^\delta(z)\) exists for all \(t \in [0,1]\) and \(z \in T_\delta M_0\), and \(\|F_t^\delta - f_t\|_{C^r(T_\delta M_t)} = o(\delta^{k-r})\), uniformly in \(t\). In fact, it follows from the proof of this lemma (see section 4 and [FL]) that the same approximation also holds for the flow from time \(t\) to time \(s\); if we let \(f_{t,s} = f_s \circ f_t^{-1}: T_\delta M_t \to C^n\) denote the flow of \(X_t\) from \(t\) to \(s\) and \(F_{t,s} = F_s \circ (F_t)^{-1}\) the flow of \(Y_t^\delta\) from \(t\) to \(s\), then for small \(\delta > 0\) the flow \(F_{t,s}^\delta\) exists for all \(s \leq t \leq 1\), and we have \(\|F_{t,s}^\delta - f_{t,s}\|_{C^r(T_\delta M_t)} = o(\delta^{k-r})\), uniformly in \(s\) and \(t\). Since \(f_{t}^{-1} = f_{t,0}\), the second estimate in theorem 1.7 follows.

Finally, if \(f_t\) is an exact \(\omega\)-flow, i.e., \(i_{f_t}^*\alpha_t\) is exact on \(M_t\) for each \(t\), and if each \(M_t\) is also polynomial convex, then by (the proof of) theorem 7.1 above we may choose \(u_t^\delta(z) = \sum_{j=1}^N x_j^\delta(t) dv_j(z)\), where \(v_j(z)\) are entire \((p-1)\)-forms on \(C^n\) and \(x_j^\delta(1 \leq j \leq N)\) are \(C^\infty\) functions with compact support in \(R\) which form a partition of unity on \([0,1]\). We may even assume that \(v_j\) are \((p-1)\)-forms with polynomial coefficients. This means that the polynomial vector fields \(X_j^\delta\) on \(C^n\), uniquely defined by the equation \(dv_j = X_j^\delta\omega\), are divergence free (resp. Hamiltonian). By proposition 4.1 in [F4] these can be written as finite sums \(X_j^\delta(z) = \sum_{k=1}^{N_{j,k}} X_{j,k}(z)\), where \(X_{j,k}\) are complete divergence free (resp. Hamiltonian) polynomial vector fields on \(C^n\) (in fact they are shear fields). Completeness means that the fields \(X_{j,k}\) may be integrated in time for all \(t \in C\) (and initial points \(z \in C^n\)). Then \(Y_{j,k}(t,z) := x_{j,k}^\delta(t)X_{j,k}(z)\) is also a complete vector field whose integral curves are reparametrizations of the integral curves of \(X_{j,k}\). Hence we may write \(Y_t^\delta = \sum_{j,k} Y_{j,k}(t,z)\), i.e., \(Y_t^\delta\) is the sum of complete, divergence free (resp. Hamiltonian), time-dependent, polynomial (in \(z \in C^n\)) vector fields. For the rest of this proof it is more convenient to write this sum as \(\sum_{t=1}^N Y_i(t,z)\), where each \(Y_i\) is one of the \(Y_{j,k}\) above.

Let \(G_{t,t+s}^i\) be the flow of \(Y_i(t,z)\) from time \(t\) to time \(t+s\). This means that \(G_{t,t+s}^i(z) = z\) and \(\frac{d}{dt} G_{t,t+s}^i(z) = Y_i(t+s,G_{t,t+s}^i(z))\). Define \(G_{t,t+s}(z) = (G_{N,t,s}^N \circ \cdots \circ G_{t,t+s}^1)(z)\). We can regard this as the flow of a time-dependent vector field \(X_{t,t+s}^\delta\), defined for times \(t'\) between \(t\) and \(t+s\); for \(t + \frac{j-1}{N}s \leq t' \leq t + \frac{j}{N}s\) we define \(X_{t,t+s}^\delta(z) = \frac{1}{N}Y_j(t+N(t' - \frac{j-1}{N}s),z)\). If we reparametrize time such that the joints are passed at zero speed, we may even assume that \(X_{t,t+s}^\delta\) is smooth and vanishes near the endpoints. We denote this smooth flow by \(G_{t,t+s}^\delta(z)\).

By definition, \(G_{t,t+s}^\delta(z) = G_{t,t+s}^\delta(z)\). Since the vector fields \(Y_j\) are complete divergence free (resp. Hamiltonian) entire vector fields, it follows that \(G_{t,t+s}^\delta\) is a holomorphic \(\omega\)-flow, i.e., \((G_{t,t+s}^\delta)^*\omega = \omega\) when \(t \leq t' \leq t + s\).
For each \( m \in \mathbb{N} \) we define the concatenations \( F^m_1(z) = (G_1 - \frac{j}{m}, \ldots \circ G_0 - \frac{j}{m})(z) \). Then by supplement 4.1.A of [AMR] we have \( \lim_{m \to \infty} F^m_1(z) = F^\delta_1(z) \), uniformly for \( z \in T_2 M_0 \). As above, we can view \( F^m_1(z) \) as the time-one map of the flow of the vector field \( X_t \) defined by \( X_t = X_t^{i-1} \frac{m}{t} \) for \( t \in \left[ \frac{i-1}{m}, \frac{j}{m} \right] \), \( 1 \leq j \leq m \). Let \( F^m(z) \) be the flow of this vector field. It is easy to see that we can arrange that \( \lim_{m \to \infty} F^m = F^\delta \), uniformly in \([0,1] \times T_2 M_0 \), and the Cauchy estimates imply \( \|F^m - F^\delta\|_{C^k(\mathcal{M}_t)} < \epsilon \) for all \( t \in [0,1] \) and all sufficiently large \( m \in \mathbb{N} \). Similarly, \( (F^m)^{-1} \) is a concatenation and hence \( \lim_{m \to \infty} (F^m)^{-1} = (F^\delta)^{-1} \) uniformly on \( T_2 M_0 \); it follows that \( \lim_{m \to \infty} (F^m)^{-1} = (F^\delta)^{-1} \) on \( M_t \), hence the result follows by the Cauchy estimates.

\[ \nw \]

Proof of Theorem 1.8. We shall see that in all cases except (iii) and (vi) the pull-back \( i^*_t \alpha_t \) of the form \( \alpha_t = X_t|\omega \) to \( M_t \) is exact for each \( t \); hence \( f_t \) is an exact \( \omega \)-flow and the result follows from the second part of theorem 1.7.

In case (i) we have \( i^*_t \alpha_t = 0 \) by degree reason. In cases (ii), (iv), (v) and (vii) we first see that the form \( i^*_t \alpha_t \) is closed on \( M_t \), either by degree reasons or by the comment after definition 2 in sect. 1; hence the cohomological assumptions imply in each of these cases that \( i^*_t \alpha_t \) is exact on \( M_t \).

For the two remaining cases (iii) and (vi) it is shown on pages 439 and 441 of [F3] that the initial family \( f_1 \) may be altered to an exact, totally real and polynomially convex \( \omega \)-flow, without changing the maps \( f_0 = Id \) and \( f_1 \); hence the result again follows from theorem 1.7.

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References.


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