Labeled $K_{2,t}$ minors in plane graphs

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ISSN 1318-4865

September 2, 1998
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Abstract

Let $G$ be a 3-connected planar graph and let $U \subseteq V(G)$. It is shown that $G$ contains a $K_{2,t}$ minor such that $t$ is large and each vertex of degree 2 in $K_{2,t}$ corresponds to some vertex of $U$ if and only if there is no small face cover of $U$. This result cannot be extended to 2-connected planar graphs.

1 Introduction

Let $G$ be a graph and $U \subseteq V(G)$. A subgraph $H$ of $G$ is called a $K_{2,t}$-preminor if it consists of pairwise disjoint trees $Z_1, Z_2$ and $T_1, \ldots, T_t$ together with edges $z_it_j$, where $z_i \in V(Z_i)$ and $t_j \in V(T_j)$, $1 \leq i \leq 2, 1 \leq j \leq t$. After contracting the edges in each of these trees, $H$ becomes the complete bipartite graph $K_{2,t}$. Clearly, $K_{2,t}$ is a minor of $G$ if and only if $G$ contains a $K_{2,t}$-preminor. If each $T_j, 1 \leq j \leq t$, contains a vertex of $U$, then $H$ is said to be $U$-labeled and we also say that $G$ contained a $U$-labeled $K_{2,t}$-minor.

Suppose now that $G$ is a 3-connected planar graph. A set $\mathcal{F}$ of facial cycles of $G$ is a face cover of $U$ if each vertex of $U$ belongs to a member of $\mathcal{F}$. The aim of this paper is to show that $G$ contains a labeled $K_{2,t}$-minor, where $t$ is large, if and only if there is no small face cover of $U$. Our original motivation for this problem came from the study of the genus of apex graphs (cf. [3]).

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‡‡Supported in part by the Ministry of Science and Technology of Slovenia, Research Project J1-0502-0101-08.
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Bienstock and Dean [1] proved that nonexistence of small face covers is closely related to the existence of large vertex packings, where by a vertex packing of $U$ we mean a subset $W$ of $U$ such that no two vertices of $W$ lie in a common facial cycle. Let $\nu(U)$ be the size of a largest packing of $U$, and let $\tau(U)$ be the size of the smallest face cover of $U$.

**Theorem 1.1 (Bienstock and Dean [1])** Let $G$ be a plane graph and $U \subseteq V(G)$. Then

$$\nu(U) \leq \tau(U) \leq 27 \nu(U).$$

As noted in [1], the constant 27 in Theorem 1.1 can be improved, and there are examples which show that it cannot be replaced by anything smaller than 2.

The main result of this paper shows that the $U$-labeled $K_{2,3}$-minors present another obstruction for small face covers in case of 3-connected planar graphs.

**Theorem 1.2** There is a nondecreasing integer function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} f(n) = \infty$ and such that the following holds. Let $G$ be a 3-connected planar graph and let $U \subseteq V(G)$. Then $G$ contains a $U$-labeled $K_{2,3}$-minor where $t \geq f(\tau(U))$. Conversely, if $G$ contains a $U$-labeled $K_{2,3}$-minor, then $\tau(U) \geq t/2$.

Theorem 1.2, whose proof is deferred to the end of Section 2, cannot be extended to the 2-connected case as the following example shows. Let $G$ be the graph composed of $n$ copies of the 4-cycle $Q_i = v_i^1v_i^2v_i^3v_i^4$ linked cyclically so that the vertex $v_i^3$ is adjacent to $v_i^{i+1}$, together with additional vertices $U = \{u_1, \ldots, u_n\}$ where $u_i$ is adjacent to all vertices of $Q_i$, $i = 1, \ldots, n$. Then $\tau(U) = \nu(U) = n$ for every embedding of $G$ in the plane. However, $G$ does not contain a $U$-labeled $K_{2,3}$-minor.

2 The proof of Theorem 1.2

We need the following definitions. Let $G$ be a graph and let $C$ be a cycle of $G$. A $C$-bridge of $G$ is either an edge $e = xy \in E(G) \setminus E(C)$ such that $x, y \in V(C)$ or a connected component of $G - V(C)$ together with all edges from this component to $C$ and all end vertices of these edges. If $B$ is a $C$-bridge, then the vertices of $V(B) \cap V(C)$ are called the vertices of attachment of $B$.

By a plane graph we mean a planar graph $G$ with a fixed embedding into the Euclidean plane. If $C$ is a cycle of a plane graph $G$, then $\text{Int}(G)$ (resp.,
Ext(G) denotes the subgraph of G formed by C and all vertices and edges inside (resp., outside) C.

It is well known that facial cycles of a 3-connected planar graph G are (precisely) the induced nonseparating cycles of G. This implies:

**Lemma 2.1** Let G be a 3-connected planar graph, let F be a facial cycle of G, and let u, v be vertices of G that do not lie on C. Then G contains a path from u to v which is disjoint from F.

Let G be a plane graph and \( C_0, \ldots, C_k \) a sequence of pairwise disjoint cycles of G such that for all indices \( i, j, 0 \leq i < j \leq k \), \( C_i \subseteq \text{Int}(C_j) \). Then we say that \( C_0, \ldots, C_k \) is a sequence of nested cycles. Let \( D_i = \text{Ext}(C_i) \cap \text{Int}(C_{i+1}) \), \( 0 \leq i < k \). If each \( D_i \) \( (1 \leq i \leq k - 2) \) contains a vertex of \( U \), then we say that \( C_0, \ldots, C_k \) are interlaced with vertices of \( U \).

**Lemma 2.2** Let G be a 3-connected plane graph and \( U \subseteq V(G) \). Suppose that \( C_0, \ldots, C_k \) is a nested sequence of cycles that are interlaced with \( U \). Then G contains a U-labeled \( K_{2,t} \)-minor where \( t = \lfloor (k - 3)/18 \rfloor \).

**Proof.** Since G is 3-connected, there exist pairwise disjoint \((C_0, C_k)\)-paths \( P_1, P_2, P_3 \). Select these paths so that the number of connected components of \( P_i \cap \{ C_0 \cup \cdots \cup C_k \} \), \( i = 1, 2, 3 \), is minimum. Let \( H = C_0 \cup P_1 \cup P_2 \cup P_3 \cup C_k \).

Suppose that \( v \in V(C_i) \setminus V(H) \) \( (1 \leq i < k) \). Since the cycles \( C_0, \ldots, C_k \) are nested, each of \( P_1, P_2, P_3 \) intersects \( C_i \). Starting at \( v \), we traverse \( C_i \) to the left and to the right until we reach one of the paths. Our choice of the paths guarantees that the path reached on the left is not the same as the one reached on the right.

Suppose that \( u \in V(D_i) \), where \( 1 \leq i \leq k - 2 \). Let \( Q \) be a path from \( u \) to a vertex in \( H \) such that only the end vertex of \( Q \) is in \( H \). (In particular, if \( u \in V(H) \), then \( Q \) is just the trivial path.) Then we say that \( Q \) joins \( u \) and \( H \). We say that \( u \) is local on \( H \) if every path which joins \( u \) and \( H \) ends on the same path \( P_j \), \( j \in \{1, 2, 3\} \). If \( Q \) ends on \( C_0 \) or \( C_k \), then it intersects \( C_i \) or \( C_{i+1} \). This implies (by the previous paragraph) that every nonlocal vertex \( u \in V(D_i) \setminus V(H) \) can be joined to two distinct paths among \( P_1, P_2, P_3 \) by using paths contained in \( D_i \).

Let \( u_i \in V(D_i) \), \( i = 1, \ldots, k - 2 \), be vertices of \( U \) which interlace with the nested cycles. If \( 6t \) of the vertices \( u_i \) are nonlocal on \( H \), then \( 2t \) of them can be joined to the same pair of the paths, say \( P_1 \) and \( P_2 \). Since \( u_i \) can be joined to \( P_1 \) and \( P_2 \) inside \( D_i \), there is a subset of \( t \) of the vertices \( u_i \) whose paths joining \( u_i \) with \( P_1 \) and \( P_2 \) are pairwise disjoint for distinct indices.
Then there is a $U$-labeled $K_{2,t}$-preminor using $P_1, P_2$, the corresponding vertices $u_i$ and the paths joining $u_i$ to $P_1$ and $P_2$ inside $D_i$. Therefore we may assume that at most $6t - 1$ vertices $u_i$ ($1 \leq i \leq k - 2$) are not local on $H$. Therefore, we may assume that at least $k/3 - 2t - 1 \geq (2k - 6)/9$ vertices $u_i$ ($2 \leq i \leq k - 3$) are local on $P_3$, say.

Suppose now that $u_i \in V(D_i)$ is local on $P_3$, where $2 \leq i \leq k - 3$. Take a path $Q_1$ joining $u_i$ with a vertex $v$ on $P_3$. Since $u_i$ is local, $Q_1 \subseteq D_i$. Let $Q_2$ be the maximal segment of $P_3$ which contains $v$ such that $Q_2$ is contained in $D_{i-1} \cup D_i \cup D_{i+1}$. Then the following holds either for $j = i$ or for $j = i + 1$: $Q_2 \cap C_j$ contains a connected component $S$ such that one of the edges of $P_3$ incident with an end of $S$ is in $D_{j-1}$ and the edge of $P_3$ incident with the other end of $S$ is in $D_j$. Going left and right on $C_j$ from $S$, we reach a path $P_e$ on the left and $P_d$ on the right where $c, d \neq 3$ by our choice of the paths. If $c = d$, then the traversed segment of $C_j$ has connected intersection $S$ with $P_3$. Therefore it does not cross $P_3$. This implies that $P_3$ reaches and leaves $S$ from the same side (either from the inside of $D_{j-1}$ or from the inside of $D_j$), a contradiction. This shows that $u_i$ can be linked to both paths $P_1$ and $P_2$ using paths inside $D_{i-1} \cup D_i \cup D_{i+1}$. Therefore, the paths for every fourth index $u_i$ (where $u_i$ is local on $P_3$) are pairwise disjoint. The number of such indices $i$ is at least $(2k - 6)/36 \geq t$. Consequently, there is a $U$-labeled $K_{2,t}$-minor which can be obtained in the same way as above. \[\Box\]

**Lemma 2.3** Let $G$ be a 2-connected graph, $U \subseteq V(G)$, and let $p, q$ be adjacent vertices of $G$. Let $t = \lceil \sqrt{|U|} \rceil$. Then either there is a cycle through the edge $pq$ which contains $t$ vertices of $U$, or $G$ contains a $U$-labeled $K_{2,t}$-minor.

**Proof.** Each ear decomposition of $G$ starting with a cycle containing the edge $pq$ determines an st-numbering $s : V(G) \rightarrow \{1, \ldots, |V(G)|\}$ with $s(p) = 1$ and $s(q) = |V(G)|$ (cf. [2]). In that numbering, every vertex distinct from $p$ and $q$ has a neighbor with a smaller number and a vertex with larger number. This gives rise to a partial order \( \preceq \) on $V(G)$ where $v \preceq u$ if there is an $s$-monotone increasing path in $G$ whose initial vertex is $v$ and terminal vertex is $u$. Consider the induced partial order on $U$. By the Dilworth Theorem, the size of a maximal antichain in this partial order is equal to the minimum number of chains covering $U$. This implies that there is either an antichain of cardinality $t$, or there is a chain containing at least $t$ elements of $U$. In the first case, the set of $s$-monotone paths from the vertices in the antichain to $q$ together with the set of $s$-monotone paths
from $p$ to these vertices contain a $U$-labeled $K_{2,2}$-minor. In the latter case, the chain gives rise to a $(p, q)$-path containing the chain. Together with the edge $pq$ we have the required cycle.

Let $C$ be a cycle of a graph $G$, and let $B$ be a $C$-bridge. Two vertices $x, y \in V(C)$ are separated by $B$ if there are vertices $a, b \in V(B) \cap (V(C) \setminus \{x, y\})$ such that they appear on $C$ in the cyclic order $a, x, b, y$. Two distinct $C$-bridges $B_1, B_2$ are separated by a $C$-bridge $B_3$ if $B_3 \neq B_1, B_2$ and there are vertices $x \in V(B_1) \cap V(C)$ and $y \in V(B_2) \cap V(C)$ such that $B_3$ separates $x$ and $y$ on $C$.

**Lemma 2.4** There is a nondecreasing integer function $g : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} g(n) = \infty$ and such that the following holds. Let $C$ be a cycle of a 2-connected plane graph $G$ and let $U = \{u_1, ..., u_k\}$ be a subset of $V(C)$ such that the vertices in $U$ appear on $C$ in the cyclic order $u_1, ..., u_k$ and no facial cycle of $\text{Int}(C)$ except $C$ contains more than one vertex in $U$. Then there is a subsequence $1 \leq i_1 < i_2 < \cdots < i_s \leq k$, where $s = g(k)$, for which one of the following holds:

(a) Let $v_j = u_{i_j}$, $1 \leq j \leq s$. Denote by $S_j$ the open segment of $C$ from $v_j$ to $v_{j+1}$, $1 \leq j \leq s$. Then there is a $C$-bridge $B$ in $\text{Int}(C)$ which has a vertex of attachment in each $S_j$, $1 \leq j \leq s$.

(b) There is an index $i_0 \in \{1, ..., s\}$ and $s - 1$ distinct $C$-bridges $B_i$, $i \in \{1, ..., s\}\setminus\{i_0\}$ such that each $B_i$ has a vertex of attachment in $S_{i_0}$ and in $S_i$.

(c) There is a facial cycle $F$ in $\text{Int}(C)$ which has a vertex in each segment $S_j$, $1 \leq j \leq s$, and does not contain any of the vertices $v_j$, $1 \leq j \leq s$.

**Proof.** A $C$-bridge $B$ is called $U$-essential if it separates two vertices in $U$. Let $T_i$ denote the open segment of $C$ from $u_i$ to $u_{i+1}$, $1 \leq i \leq k$. If $B$ is a $C$-bridge, $I(B)$ denotes the set of all indices $i$ such that $T_i$ contains a vertex of attachment of $B$. Obviously, a $C$-bridge $B$ is essential if and only if $|I(B)| > 1$. A $C$-bridge $B_1$ covers a $C$-bridge $B_2$ if $I(B_1) \supseteq I(B_2)$. Let $\mathcal{B}$ denote a minimal set of $U$-essential $C$-bridges such that every $U$-essential $C$-bridge is covered by one in $\mathcal{B}$, and let $d = \max\{|I(B)| : |B| \in \mathcal{B}\}$. Since no two vertices in $U$ belong to the same facial cycle of $\text{Int}(C)$ distinct from $C$, each $T_i$, $1 \leq i \leq k$, contains a vertex of attachment of some $C$-bridge in $\mathcal{B}$. Consequently, $d|\mathcal{B}| \geq k$.

Let $\mathcal{A} \subseteq \mathcal{B}$ be a largest set of $C$-bridges such that no two $C$-bridges in $\mathcal{A}$ are separated by a $C$-bridge in $\mathcal{B}$ and let $l = |\mathcal{A}|$. Then it is not
hard to see, that no two $C$-bridges in $A$ are separated by any $C$-bridge of $\text{Int}(C)$. Consequently, there is a facial cycle $F$ of $\text{Int}(C)$ such that $F \neq C$ and $F$ contains at least two vertices of attachment of each $C$-bridge in $A$.

Since any $C$-bridge in $A$ separates two vertices in $U$ there is a subsequence $1 \leq i_1 < \cdots < i_l \leq k$ such that $F$ has a vertex in each segment of $C$ from $u_{i_j}$ to $u_{i_{j+1}}$ and $F$ does not contain any vertex $u_{i_j}, 1 \leq j \leq l$.

Let $B_1$ denote the set of all $C$-bridges $B \in B$ such that $B$ does not separate any two $C$-bridges in $B$, and for $i \geq 2$, let $B_i$ be the set of all $C$-bridges $B \in B \setminus \bigcup_{j=1}^{i-1} B_j$ such that $B$ does not separate any two $C$-bridges in $B \in B \setminus \bigcup_{j=1}^{i-1} B_j$. Let $e$ denote the largest integer such that $B_e \neq \emptyset$. A simple induction on $e$ shows that, after possibly changing the indices, there is a subsequence $1 \leq i_1 < \cdots < i_{e+1} \leq k$ and a subset $\{B_1, \ldots, B_{e-1}\}$ of $B$ such that $B_j \in B_j$ and each $B_j$ has a vertex of attachment in the open segment of $C$ from $u_{i_j}$ to $u_{i_{j+1}}$ and one in the open segment of $C$ from $u_{i_j}$ to $u_{i_{j+1}}$.

Now we wish to prove that $|B|$ is bounded by a function of $d$, $l$ and $e$. Obviously, $|B| = |B_1| + \cdots + |B_e|$. Since no two $C$-bridges in $B_e$ are separated by any other $C$-bridge, $|B_e| \leq l$. Let $1 \leq i < e$, and call two $C$-bridges in $B_i$ similar if they are not separated by any $C$-bridge in $B_{i+1} \cup \cdots \cup B_e$. It is not hard to see, that similarity is an equivalence relation on $B_i$. It follows from the definition of $B_i$, that no two similar $C$-bridges in $B_i$ are separated by any other $C$-bridge of $\text{Int}(C)$. Consequently, an equivalence class with respect to similarity consists of at most $l$ $C$-bridges. There are at most $d \sum_{j=i+1}^{e} |B_j|$ pairwise nonsimilar $C$-bridges in $B_i$ (if $i < e$). A simple inductive proof shows that

$$|B_i| \leq l(ld + 1)^{e-i}.$$  

This implies

$$|B| \leq \sum_{i=1}^{e} l(ld + 1)^{e-i} \leq \frac{1}{d}(ld + 1)^e.$$  

Let $s$ be an integer such that for every subsequence $1 \leq i_1 < i_2 < \cdots < i_s \leq k$, none of (a), (b) and (c) holds. Then $s > \max\{d, l, e\}$ and, consequently,

$$k \leq s|B| \leq z(s) = s^{2s}. \quad (1)$$  

Since $z(s)$ is an increasing function this proves the lemma. \qed

Let us observe that (1) shows that the function $g(k)$ in Lemma 2.4 is of order $\log(k)/\log\log(k)$. 

6
Lemma 2.5 There is a nondecreasing integer function $h : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} h(n) = \infty$ and such that the following holds. Let $C$ be a cycle of an arbitrary 3-connected plane graph $G$ and let $u_1, \ldots, u_k$ be vertices which appear on $C$ in that order such that no two of them belong to the same facial cycle. Then $G$ contains a $\{u_1, \ldots, u_k\}$-labeled $K_{2,t}$-minor, where $t = h(k)$.

Proof. By Lemma 2.4, there is a subsequence $v_1, \ldots, v_{\log k}$ of $u_1, \ldots, u_k$ satisfying one of the cases (a)-(c) of that lemma. Repeating the same in Ext($C$) with vertices $v_1, \ldots, v_{\log k}$, we get a subsequence $z_1, \ldots, z_r, r = g(g(k))$, such that in each of Int($C$) and Ext($C$), one of the cases (a)-(c) occurs. Considering Int($C$), we denote by $w_j$ a vertex of attachment of $B$ (or a vertex of $B_j$, or a vertex of $F$ in cases (b) and (c), respectively) which belongs to the segment $S_j$, $j = 1, \ldots, r$. For Ext($C$), we denote the corresponding bridge(s) or face by $B'$ or $B_j'$, or $F'$ (respectively), and define corresponding vertices $w_j'$ on $S_j$. If case (b) occurs in Int($C$) (resp., Ext($C$)) we denote by $i_0$ (resp., $i_0'$) the index $i_0$ from Lemma 2.4. Because of symmetry, we distinguish 6 cases. We will use the notation (b|c) to denote the case where (b) occurs in Int($C$) and (c) in Ext($C$), and similarly for the other cases.

Case (a|a): Let $Z_1$ be a spanning tree in $B - V(C)$ together with an edge joining this tree with $w_j$ for each odd index $j$. Similarly, let $Z_2$ be a spanning tree in $B' - V(C)$ together with an edge joining this tree with $w_j'$ for each even index $j$. Now we get a $U$-labeled $K_{2,1}$-preminor in $G$, where $t = \lfloor r/2 \rfloor$, by adding segments of $C$ joining vertices $w_j$ and $w_j'$, $j = 1, 3, 5, \ldots$.

Case (a|b): This case is similar to the above, except that the tree $Z_2$ is obtained as follows. We may assume that $i_0 = r$. Now, start with spanning trees in interiors of bridges $B_j' - V(C)$ together with edges from these trees to $w_j'$, $j = 2, 4, 6, \ldots$. Finally, add the segment $S_j$ and edges from these trees to $S_j$. Then we get a $U$-labeled $K_{2,1}$-minor in $G$, where $t = \lfloor (r - 1)/2 \rfloor$.

Case (a|c): For $i \in \{1, \ldots, r\}$, let $\alpha'$ be the vertex of $F' \cap S_{i-1}$ which is closest to $u_i$ on $C$. Similarly, let $\beta'$ be the vertex of $F' \cap S_i$ as close as possible to $u_i$ on $C$. Then the segment $\alpha'$ of $F'$ from $\alpha'$ to $\beta'$ is internally disjoint from $C$. Let $\alpha$ and $\beta$ be attachments of $B$ on $S_{i-1}$ and $S_i$, respectively, chosen as close as possible to $u_i$. Then there is a facial cycle $R$ in Int($C$) which contains an edge $e$ of $B$ incident with $\alpha$ and contains an edge $f$ of $B$ incident with $\beta$. Let $R_B \subseteq B$ be the segment of $R$ from $e$ to $f$.

By Lemma 2.1, there is a path joining $u_i$ and $B - V(C)$ which is disjoint from $F'$. It is easy to see that such a path $P_i$ can be chosen so that it is contained in the disk bounded by $\alpha', R_B$, and the segments of $C$ joining $\alpha, \alpha'$ and $\beta, \beta'$. In particular, the paths $P_i, P_j$ are internally disjoint if $|i - j| \geq 2$.

Let $R_i$ be the union of $P_i$ and the segment of $C$ from $\alpha'$ to $\beta$. Let $T_i$ be
a spanning tree in $R_i - (V(B) \cup \{\alpha', \beta'\})$, let $e_i$ be the edge of $P_i$ connecting $T_i$ with $B - V(C)$, and let $f_i$ be an edge of $C$ joining $T_i$ and $F' = w_i'$. Now, we get a $U$-labeled $K_{2, t}$-preminor in $G$, $t = \lfloor r/2 \rfloor$, by taking a spanning tree $Z_1$ in $B - V(C)$, the path $Z_2 = C - w_i'$, the trees $T_i$ and the connecting edges $e_i, f_i$, $i = 1, 3, 5, \ldots$.

Case (b|c): This case is similar to the case (a|c) above except that we consider the union of $S_i$ and the bridges $B_{1}, \ldots, B_{r-1}$ to play the role of the bridge $B$.

Case (b|b): We assume that $i_0 = r$. If $i_0 \neq i_0'$, we can proceed similarly to the case (a|b) above except that we consider the union of $S_i$ and the bridges $B_{i}, i \in \{1, \ldots, r\} \setminus \{i_0'\}$ to play the role of the bridge $B'$. Thus we may assume that $i_0 = i_0' = r$.

Let $q = \lceil (r - 1)^{1/3} \rceil$. Let $z_j (z_j')$ be a vertex of $B_j$ (resp. $B_j'$) in $S_r$. If $x, y \in V(S_r)$, we write $x \preceq y$ if $x$ is closer to $u_0$ on $S$, than $y$. Clearly, $z_1 \preceq z_2 \preceq \cdots \preceq z_{r-1}$ and $z_1' \preceq z_2' \preceq \cdots \preceq z_{r-1}'$. We distinguish three subcases.

(i) There is an index $i$ such that $z_i = z_i' = \cdots = z_{i+q} = z_{i+q}'$. In this case we remove all edges of $B_{i+1}, \ldots, B_{i+q-1}$ incident with $z_i$. The resulting graph $G'$ is 2-connected (since $G - z_i$ is 2-connected). Let $F'$ be the new facial cycle of $G'$. Now, a proof similar to the case (a|c) shows that there is a $U$-labeled $K_{2, t}$-minor, where $t = \lfloor (q - 3)/2 \rfloor$.

(ii) There is an index $i$ such that $z_{i+q} < z_i'$. This case is similar to the case (a|a) where the union of the segment of $S_r$ from $z_i$ to $z_{i+q}$ and bridges $B_i, \ldots, B_{i+q}$ play the role of $B$, while the union of the segment of $S_r$ from $z_i'$ to $z_{i+q}'$ and bridges $B_i', \ldots, B_{i+q}'$ play the role of $B'$.

A similar proof works if $z_{i+q}' < z_i$.

(iii) Otherwise: In this case, there are indices $1 \leq i_1 < i_2 < \cdots < i_q \leq r$ such that for $j = 1, \ldots, q$ we have $z_{i_j} < z_{i_j+q} < z_{i_j+4} < z_{i_j+6}$. Let $Q_j$ be a cycle contained in $B_{i_j} \cup B_{i_j}^c \cup S_{i_j}$ and the segment of $S_r$ from $z_{i_j}$ to $z_{i_j}'$. The cycles for $j = 1, 5, 9, \ldots$ are pairwise disjoint and nested and are interlaced with $U$. Now, Lemma 2.2 applies.

Case (c|c): Since $G$ is 3-connected, $F \cap F'$ is connected. Therefore, we may assume that $F \cap F' \subseteq S_r$. Suppose that $3 \leq i \leq r - 2$. By Lemma 2.1, there is a path $P_i$ joining $u_i$ and $F$ which is disjoint from $F'$. Let $A'$ be the segment of $F'$ defined as in case (a|c), and let $A$ be the segment of $F$ defined in the same way. It is easy to see that we may assume that $P_i$ is
contained in the disk bounded by $A \cup A'$ and the two segments of $C$ joining the ends of $A$ and $A'$.

We define similarly $P_i$, a path joining $u_i$ and $F'$ which is contained in the same disk as $P_i$ and is disjoint from $F$. Now, we have a $U$-labeled $K_{2,1}$-minor, $t = [(r - 4)/2]$, in the similar way as in previous cases, where we take $Z_i = F - S_i$, $Z_2 = F' - S_i$, and $T_i$ a spanning tree in $(P_i \cup P_i') - (F \cup F')$, $i = 3, 5, 7, \ldots$.

\textbf{Proof.} (Of Theorem 1.2). Let $U'$ be a subset of $U$ such that $|U'| = \tau(U)$ and no two vertices in $U'$ belong to the same facial cycle of $G$. Furthermore, let $t = \lfloor \sqrt{|U'|} \rfloor$. By Lemma 2.3 either $G$ contains a $U$-labeled $K_{2,1}$-minor or there is a cycle $C$ of $G$ which contains $t$ vertices of $U'$. In the latter case it follows from Lemma 2.5 that $G$ contains a $U$-labeled $K_{2,t(\tau(U))}$-minor. This proves the first part of Theorem 1.2.

To prove the second part, let $H$ be a plane embedding of a $K_{2,1}$ and let $A, B$ denote the color classes of $H$ such that $|A| = t$ and $|B| = 2$. For an arbitrary embedding of $H$ in the plane, every facial cycle of $H$ has length four and contains precisely two vertices in each color class. Consequently, each face cover of $A$ in $H$ contains at least $t/2$ facial cycles. It follows that if $G$ contains a $U$-labeled $K_{2,1}$-minor, then $\tau(U) \geq t/2$.

It is worth mentioning that the proof of Theorem 1.2 is constructive and yields a polynomial time algorithm such that, given a 3-connected planar graph $G$ and vertex set $U \subseteq V(G)$, no two vertices of which are in the same facial cycle, finds a $U$-labeled $K_{2,\lfloor|U|\rfloor}$-preminor in $G$.

\textbf{References}

