WEAKLY FLAG-TRANSITIVE CONFIGURATIONS
AND HALF-ARC-TRANSITIVE GRAPHS

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Abstract

A configuration is weakly flag-transitive if its group of automorphisms acts transitively on flags but the group of all automorphisms and anti-automorphisms acts transitively on flags. It is shown that weakly flag-transitive configurations are in one-to-one correspondence with bipartite \( \frac{1}{2} \)-arc-transitive graphs of girth not less than 6. Several infinite families of weakly flag-transitive configurations are given via their Levi graphs. Among others an infinite family of non-self-polar weakly flag-transitive configurations is constructed. The smallest known weakly flag-transitive configuration has 27 points and the smallest known non-self-polar weakly flag-transitive configuration has 34 points.

1 Introduction

The topic studied in this paper touches theories of graphs, groups and configurations and the reader is thus referred to [1, 5, 8, 10, 14, 16] for the terms not defined here. Unless specified otherwise, all objects considered in this paper are assumed to be finite. Moreover, all graphs are simple and undirected.

By a configuration we shall always mean a symmetric configuration. More precisely, an \( n_k \)-configuration is an ordered triple \( C = (P, B, I) \) of mutually

disjoint sets $P$, $B$ and $I$ (whose elements are called, respectively, points, blocks (lines) and flags) with $|P| = |B| = n$ and $I \subseteq P \times B$, and (with a point $p$ and a block $B$ called incident if $(p, B) \in I$) such that each block (point) is incident with the same number $k$ of points (blocks), and two distinct points (blocks) are incident with at most one common block (point). To each $n_k$-configuration $C = (P, B, I)$ the dual $n_k$-configuration $C^* = (B, P, I^*)$ may be associated in the usual way by reversing the roles of points and blocks in $C$.

An automorphism of a configuration $C$ is an incidence preserving permutation on the (disjoint) union $P \cup B$ which maps $P$ to $P$. Similarly, an anti-automorphism of a configuration $C$ is an incidence preserving permutation on the (disjoint) union $P \cup B$ which interchanges $P$ and $B$. The configuration $C$ is said to be self-dual if it admits an anti-automorphism, that is, if it is isomorphic with its dual $C^*$. An anti-automorphism of order 2 is called a polarity. We say that $C$ is self-polar if it admits a polarity. We let $\text{Aut}_0C$ denote the group of all automorphisms of $C$, and we let $\text{Aut}C$ denote the group of all automorphisms and anti-automorphisms of $C$. Then $\text{Aut}_0C$ is a proper subgroup of $\text{Aut}C$ (of index 2) if $C$ is self-dual, and coincides with $\text{Aut}C$ otherwise.

If $X$ is a graph we let $V(X)$ and $E(X)$ denote the respective sets of vertices and edges. For $v_1, \ldots, v_k \in V(X)$ and a positive integer $d$ we let $N^d(v_1, \ldots, v_k)$ denote the set of all vertices in $X$ at distance $d$ from the set $\{v_1, \ldots, v_k\}$. In particular, $N(v_1, \ldots, v_k) = N^1(v_1, \ldots, v_k)$ is the set of neighbors of $\{v_1, \ldots, v_k\}$. Furthermore, by $\text{Aut}X$ we denote the automorphism group of $X$. For a bipartite graph $X$ we let $\text{Aut}_0X$ be the subgroup of $\text{Aut}X$ fixing the bipartition.

The concept of a Levi graph of a configuration was introduced by Coxeter in 1950 (see [7]). Given a configuration $C = (P, B, I)$ we let $L(C) = L(P, B, I)$ be the bipartite graph with “black” vertices $P$ and “white” vertices $B$ and with an edge joining some $p \in P$ and some $B \in B$ if and only if $(p, B) \in I$. Note that dual configurations have the same Levi graph with the roles of black and white vertices interchanged. Clearly, a complete information about the configuration can be recovered from its Levi graph with a given black and white coloring of vertices. The following proposition from [7] characterizes $n_k$-configurations in terms of their Levi graphs.

**Proposition 1.1** A graph $X$ is a Levi graph of some $n_k$-configuration if and only if is a regular bipartite graph of valency $k$ with girth at least 6.

In view of Proposition 1.1 we may call a configuration connected if and
only if its Levi graph is connected. In this paper we consider, unless explicitly stated otherwise, only connected graphs and configurations.

A graph \(X\) is said to be vertex-transitive, edge-transitive and arc-transitive, respectively, if its automorphism group \(\text{Aut} X\) transitively on the sets of its vertices, edges and arcs. Further, we shall say that \(X\) is \(\frac{1}{2}\)-arc-transitive provided it is vertex- and edge- but not arc-transitive. More generally, by a \(\frac{1}{2}\)-arc-transitive action of a subgroup \(G \leq \text{Aut} X\) on \(X\) we shall mean a vertex- and edge- but not arc-transitive action of \(G\) on \(X\). In this case we shall say that the graph \(X\) is \((G, \frac{1}{2})\)-transitive. For further references on \(\frac{1}{2}\)-arc-transitive graphs (also \(\frac{1}{2}\)-transitive graphs) see [6, 12, 13] and the survey paper [11].

Given a configuration \(C = (P,B,F)\), we shall, for the purpose of this paper, make no distinction between the two groups \(\text{Aut} C\) and \(\text{Aut} L(C)\), as well as the two groups \(\text{Aut}_0 C\) and \(\text{Aut}_0 L(C)\). We say that \(C\) is point-, block-, and flag-transitive provided \(\text{Aut} C\) acts transitively on the sets of its points, blocks and flags, respectively. Moreover, a flag-transitive configuration \(C\) is weakly flag-transitive if \(\text{Aut} C\) acts intransitively on the sets of its flags, and is strongly flag-transitive otherwise.

For each \(n\) there is only one \(n_2\)-configuration, called the \(n\)-gon; in particular, the configuration \(3_2\) is called the triangle. The \(n\)-gon is a self-dual, point-transitive, block-transitive and strongly flag-transitive configuration whose Levi graph is the cycle \(C_{2n}\). As for \(n_3\)-configurations, the situation is much more complex since their number grows very fast with growing \(n\) (see [3]). Many historically important configurations are of this kind and have received considerable attention over the years. Among others, an important problem is the question of realizability of such configurations (see [4]). However, no \(n_3\)-configuration is weakly flag-transitive, making \(n_4\)-configurations the simplest candidates for weak flag-transitivity. This is the primary motivation for this work. We explore the connection between weakly flag-transitive configurations and \(\frac{1}{2}\)-arc-transitive graphs. Using this connection we give several new families of weakly flag-transitive configurations with certain prescribed properties.

In Section 2 we show that weakly flag-transitive configurations are in one-to-one correspondence with bipartite \(\frac{1}{2}\)-arc-transitive graphs of girth at least 6 (Theorem 2.6). As a consequence, there are no weakly flag-transitive \(n_k\)-configurations, for \(k\) odd (Corollary 2.7). Some structural properties of weakly flag-transitive configurations, such as the concept of a kernel, are also discussed there.
Section 3 is devoted to specific constructions of weakly flag-transitive configurations. In view of Bouwer’s construction of a family of bipartite \(\frac{1}{2}\)\text{-}arc-transitive graph of girth 6 for every even valency greater than 2 (see [6]), it follows that there are weakly flag-transitive \(n_k\)-configurations for every even \(k > 2\) (see Proposition 3.1). Several constructions of weakly flag-transitive \(n_k\)-configuration whose kernels are odd length polygons are known [12]. In Theorem 3.3 we give a construction of weakly flag-transitive \(n_4\)-configurations whose kernels are even length polygons. Note that all weakly flag-transitive configurations are self-dual. An infinite family of non-self-polar, weakly flag-transitive \(n_4\)-configurations is given in Theorem 3.4.

2 Weakly flag-transitive configurations and \(\frac{1}{2}\)\text{-}arc-transitive graphs

To a graph \(X\) of valency \(2k\), and a subgroup \(G\) of \(\text{Aut} X\), acting \(\frac{1}{2}\)\text{-}arc-transitively on \(X\), two oppositely oriented graphs may be associated in a natural way. Let \(D_G(X)\) be one of these two oriented graphs, fixed from now on. We perform the following operation on \(D_G(X)\). Each vertex \(v\) splits into two vertices: \(v^+\) which keeps all incoming arcs and \(v^-\) which keeps all outgoing arcs. A possibly disconnected \(k\)-valent oriented graph of order \(2|V(X)|\) is thus obtained. Let \(X_G^+\) denote its underlying undirected graph. (Note that \(X_G^+\) depends solely on \(X\) and \(G\).) Let \(V_G^+ = \{v^+ : v \in V(X)\}\) and \(V_G^- = \{v^- : v \in V(X)\}\). For the special case \(G = \text{Aut} X\) the symbol \(G\) is omitted in all of the above notations.

**Lemma 2.1** Let \(X\) be a \((G, \frac{1}{2})\)-transitive graph for a subgroup \(G\) of \(\text{Aut} X\). Then the graph \(X_G^+\) is bipartite with bipartition \(\{V_G^+, V_G^-\}\), edge-transitive and has isomorphic connected components. Moreover, \(\text{Aut} X_G^+\) contains an isomorphic copy of \(G\), acting transitively on \(E(X_G^+)^+\) and intransitively on \(V(X_G^+)\) with orbits \(V_G^+\) and \(V_G^-\).

**Proof.** Since \(V_G^+ = \{v^+ : v \in V(X_G)\}\) and \(V_G^- = \{v^- : v \in V(X_G)\}\) are independent sets, \(X_G^+\) is bipartite. The action of \(G\) extends to \(X_G^+\) in a natural way by letting \(\alpha(u^+) = \alpha(u)^+\) and \(\alpha(u^-) = \alpha(u)^-\) for each \(u \in V(X)\) and each \(\alpha \in \text{Aut} X\). Hence \(V_G^+\) and \(V_G^-\) are the orbits of the copy of \(G\) in \(\text{Aut} X_G^+\).
while edge-transitivity of its action follows from the edge-transitivity of the action of $G$ on $X$. Clearly, all components of $X_G^*$ are isomorphic graphs.

Observe that for any component $C$ of $X_G^*$ the constituent $G^V(C)$ acts transitively on $V(C)$. Furthermore, in view of the nature of the operation on $D_G(X)$ which produces $X_G^*$, there are isomorphic copies of the above components of $X_G^*$ inside $X$. Any such component (or its isomorphic counterpart in $X$) will be called a $G$-kernel of $X$ and will be denoted by $\text{Ker}_G X$. As above, for the special case $G = \text{Aut} X$ the symbol $G$ is omitted. The proof of the proposition below, which summarizes some basic properties of $\text{Ker}_G X$, is left to the reader.

**Proposition 2.2** Let $X$ be $(G,\frac{k}{2})$-transitive graph of valency $2k$ with $G \leq \text{Aut} X$. Then the following statements hold.

(i) $\text{Ker}_G X$ is a $k$-valent, bipartite graph admitting an edge-transitive action of the corresponding constituent of $G$.

(ii) Either $X$ has two $G$-kernels, both spanning $X$, or $X$ has at least three $G$-kernels, which are all induced subgraphs; in both cases the collection of $G$-kernels gives rise to a decomposition of $E(X)$.

(iii) The edge sets of $G$-kernels are blocks of imprimitivity of $G$ in its action on $E(X)$.

**Proof.** The proof of (i) is an immediate consequence of Lemma 2.1.

As for part (ii), the fact that the edge sets of the $G$-kernels decompose $E(X)$, is obvious. Furthermore, note that there are two $G$-kernels containing a given vertex $v \in V(X)$: call them $\text{Ker}_G^+(v, X)$ and $\text{Ker}_G^-(v, X)$, the superscripts $+$ and $-$ reflecting the relative orientation of the arcs at $v$. More precisely $\text{Ker}_G^+(v, X)$ is determined by all $G$-alternating paths in $D_G(X)$ originating from $v$ while $\text{Ker}_G^-(v, X)$ is determined by all $G$-alternating paths in $D_G(X)$ terminating at $v$. Assume that the $G$-kernels are not induced subgraphs. Call an edge of $X$ *major* if it belongs to the edge set of some $G$-kernel, that is, to $E(\text{Ker}_G^+(v, X))$ for some $v \in V(X)$ and *minor* if it belongs to the edge set of the subgraph induced by the set of vertices of some $G$-kernel.

Note that there is precisely one $G$-kernel in which a given edge is major. But, in view of the fact that there are precisely two $G$-kernels containing a given vertex, it follows that there is also precisely one $G$-kernel in which
this same edge is minor. For transitivity reasons each induced subgraph on a
given $G$-kernel contains the same number of minor edges (and by definition
also the same number of major edges). But then these two numbers must be
the same and equal to $|E(X)|/|E(\ker G_X)|$. In this case the graph consists of
an edge disjoint union of two $G$-kernels, both a spanning subgraph of $X$.

Part (iii) is clear. \hfill \blacksquare

The proof of the first lemma below is straightforward. The second lemma
is an extension of a classical result from [9, Theorem 1].

**Lemma 2.3** Let $G$ be a transitive permutation group on a finite set $V$ and
let $G_0$ be an intransitive subgroup of $G$ of index 2. Then $G_0$ has two orbits
on $V$ of equal size and each element of $G \setminus G_0$ interchanges these two orbits.

**Lemma 2.4** Let $X$ be a connected graph and $G$ a subgroup of $\text{Aut} X$ acting
transitively on $E(X)$ and intransitively on $V(X)$. Then $X$ is bipartite and
the two parts of bipartition are orbits of the action of $G$ on $V(X)$.

The following lemma is a straightforward consequence of Lemmas 2.1 and
2.3.

**Lemma 2.5** Let $X$ be a $(G, \frac{1}{2})$-transitive and bipartite graph with bipartition
$(U, W)$ and let $D_G(X)$ be one of the two oriented graphs associated with the
action of $G$. Color each arc with tail in $U$ green and each arc with tail
in $W$ red. Then the red and green subgraphs of $X$ consist of components,
isomorphic to $\ker G_X$.

The next result links weakly flag-transitive configurations to bipartite
$\frac{1}{2}$-arc-transitive graphs of girth at least 6.

**Theorem 2.6** A configuration $C = (P, B)$ is weakly flag-transitive if and
only if its Levi graph $L(C)$ is a bipartite $\frac{1}{2}$-arc-transitive graph of girth at
least 6.

**Proof.** Observe that $X = L(C)$ is by Proposition 1.1 a bipartite graph
with girth at least 6. Assume that $X$ is $\frac{1}{2}$-arc-transitive. Then $\text{Aut} X$ acts
transitively on $V(X)$ and $E(X)$ and moreover $\text{Auto} X$ is a subgroup of index
2 in $\text{Aut} X$ acting intransitively on $V(X)$ and $E(X)$. In the language of
configurations, $\text{Aut} C$ acts transitively on flags while $\text{Aut}_0 C$ acts intransitively on flags. Hence $C$ is weakly flag-transitive.

Suppose now that $C$ is a weakly flag-transitive configuration. By Proposition 1.1 its Levi graph $X = L(C)$ is bipartite of girth at least 6. Since $\text{Aut} X$ acts transitively on $E(X)$, whereas $\text{Aut}_0 X$ acts intransitively on $E(X)$, we see that $\text{Aut}_0 X$ is a proper subgroup of $\text{Aut} X$. But $X$ is connected and so $[\text{Aut} X : \text{Aut}_0 X] = 2$. Furthermore $\text{Aut} X$ acts transitively on $V(X)$, for otherwise the two parts $P$ and $B$ of the bipartition would, by Lemma 2.4, coincide with the two orbits of $\text{Aut} X$, forcing $\text{Aut} X = \text{Aut}_0 X$. We conclude that $X$ is vertex- and edge-transitive. Also, $\text{Aut}_0 X$ is a subgroup of index 2 in $\text{Aut} X$ acting intransitively on $V(X)$ as well as on $E(X)$. Applying Lemma 2.3 for the action of $\text{Aut}_0 X$ on $V(X)$, we see that $\text{Aut}_0 X$ has two orbits on $V(X)$, namely the sets $P$ and $B$. Similarly, applying Lemma 2.3 for the action of $\text{Aut}_0 X$ on $E(X)$ we conclude that $\text{Aut}_0 X$ has two equal-size orbits on $E(X)$ (corresponding to the “red” and “green” edges of Lemma 2.5). But since $P$ and $B$ are orbits of $\text{Aut}_0 X$ in its action on $V(X)$ it follows that every vertex of $X$ is incident with the same number of red and green edges. Orienting the red edges from $P$ to $B$ and the green edges from $B$ to $P$, we see, by Lemma 2.3, that $\text{Aut} X$ preserves this orientation. We conclude that $X$ is $\frac{1}{2}$-arc-transitive. \qed

The proof of the result below follows from the well known fact that $\frac{1}{2}$-arc-transitive graphs have even valency (see [15]), but it may also be deduced from Lemma 2.5.

Corollary 2.7 There is no weakly flag-transitive $n_k$-configuration for $k$ odd.

We may extend the concept of a kernel to weakly flag-transitive configurations in the following way. Given a weakly flag-transitive $n_{2k}$-configuration $C$ it follows from Proposition 2.2 that its Levi graph $L(C)$ is an edge-disjoint union of isomorphic copies of $\text{Ker} L(C)$, the latter being connected, bipartite, of girth at least 6, and such that $\text{Aut}_0 \text{Ker} L(C)$ acts transitively on the set of its edges. We let the kernel $\text{Ker} C$ of $C$ be, up to duality, the strongly flag-transitive $r_k$-configuration associated with $\text{Ker} L(C)$. In the particular case of an $n_4$-configuration $C$ the kernel is always an $r$-gon for some $r \geq 3$. The corresponding $2r$-cycles in the $\frac{1}{2}$-arc-transitive graph $L(C)$ are referred to as alternating cycles and the parameter $r$ is referred to as the radius of $L(C)$ (see [12]).
A natural question arises. Which strongly flag-transitive configurations can be kernels of weakly flag-transitive configurations? The known constructions of $\frac{1}{2}$-arc-transitive graphs of valency 4 for odd radius greater than or equal to 7 (see [12]) together with the constructions of such graphs for certain even radii given in Section 3, suggest that, for every $r \geq 3$, a weakly flag-transitive $n_4$-configuration whose kernel is the $r$-gon may exist. As for kernels which are strongly flag-transitive $r_k$-configurations, where $k \geq 3$, practically nothing seems to be known. In particular, it would be interesting to know if there exists a weakly-transitive configuration with a non-self-dual kernel. (Note that a weakly flag-transitive configuration is by definition self-dual. But this may not necessarily be the case for its kernel.)

The next section is devoted to the study of weakly flag-transitive $n_4$-configurations. We give a construction of an infinite family of weakly flag-transitive configurations whose kernels are even length polygons and a construction of non-self-polar weakly flag-transitive configurations.

### 3 Constructions of weakly flag-transitive $n_4$-configurations with prescribed properties

In 1970 Bouwer [6, Proposition 2] gave a construction of a bipartite $\frac{1}{2}$-arc-transitive graph of girth 6 and valency $2k$ for every $k \geq 2$. As a consequence we have the following result.

**Proposition 3.1 (Bouwer)** There exist weakly flag-transitive $n_{2k}$-configurations for every even $k \geq 2$.

The smallest graph in Bouwer’s family has 54 vertices and it gives rise to a weakly flag-transitive $27_4$-configuration which is the smallest known weakly flag-transitive configuration. As mentioned in Section 2, the kernel of an $n_4$-configuration is either an even length or an odd length polygon. For example, the kernel of the above $27_4$-configuration is the 9-gon. In fact this configuration belongs to an infinite family of weakly flag-transitive $(2m+1)_4$-configurations which have the property that any two adjacent kernels, that is, $(2m+1)$-gons with either a common point or a common line, have respectively all points or all lines in common. The corresponding Levi graphs belong to a family of graphs defined below.
Let $t \geq 3$ be an integer, $r \geq 3$ be an odd integer and let $s \in \mathbb{Z}$ satisfy $s^t = \pm 1$. The graph $X(s; t, r)$ is defined to have the vertex set $\{v_i^j : i \in \mathbb{Z}, j \in \mathbb{Z}_r\}$ and edges of the form $v_i^j v_{i+1}^{j+s}, v_i^j v_{i+1}^{-s}, (i \in \mathbb{Z}, j \in \mathbb{Z}_r)$. It is easily checked that the permutations $\rho, \sigma$ and $\tau$ mapping according to the rules
\begin{align*}
v_i^j \rho &= v_i^{j+1}, \quad i \in \mathbb{Z}, j \in \mathbb{Z}_r. \quad (1) \\
v_i^j \sigma &= v_i^{j-1}, \quad i \in \mathbb{Z}, j \in \mathbb{Z}_r. \quad (2) \\
v_i^j \tau &= v_i^{-j}, \quad i \in \mathbb{Z}, j \in \mathbb{Z}_r. \quad (3)
\end{align*}
are automorphisms of $X(s; t, r)$.

The following result from [12] classifies $\frac{1}{2}$-arc-transitive graphs among the graphs $X(s; t, r)$.

**Theorem 3.2** [12, Theorem 3.4] The graph $X \cong X(s; t, r)$, where $r \geq 3$ is odd, $t \geq 3$ and $s \in \mathbb{Z}$ satisfies $s^t = \pm 1$, is $\frac{1}{2}$-arc-transitive if and only if none of the following conditions is fulfilled.

(i) $s^2 = \pm 1$;

(ii) $(s; t, r) = (2; 3, 7)$;

(iii) $(s; t, r) = (s; 6, 7k)$, where $k \geq 1$ is odd, $(7, k) = 1$, $s^6 = 1$, and there exists a unique solution $q \in \{s, -s, 1/s, -1/s\}$ of the equation $x^2 + x - 2 = 0$ such that $7(q - 1) = 0$ and $q \equiv 5 \pmod{7}$.

In particular, when $X$ is $\frac{1}{2}$-arc-transitive then $\text{Aut} X = \langle \rho, \sigma, \tau \rangle$.

Clearly, if $t = 2u$ even, the graph $X(s; t, r)$ is bipartite and moreover, if $s \neq 1, -1$, the girth of $X(s; t, r)$ is either 6 or 8. Let $C(s; t, r)$ denote the $(ur)$-configuration corresponding to the graph $X(s; t, r)$ under these assumptions. Note that the kernel of $C(s; t, r)$ is the $r$-gon. It then follows by Theorem 3.2 above that for each odd $r \geq 7$ there exist a pair $(s, t)$ such that the configuration $C(s; t, r)$ is weakly flag-transitive, giving us weakly flag-transitive configurations whose kernels are polygons of length greater than or equal to 7. In particular, $X(2; 6, 9)$ is isomorphic to the above mentioned
smallest graph of Bouwer [6], and so \( C(2; 6, 9) \) is the above mentioned smallest known weakly flag-transitive configuration. This leaves 3-gons and 5-gons as kernels as the only open question in the case of weakly flag-transitive \( n_4 \)-configurations whose kernels are odd length polygons. This brings us to the existence of weakly flag-transitive configurations whose kernels are even length polygons. We give below an infinite family of such configurations by constructing the corresponding Levi graphs.

Let \( r, t \geq 4 \), be even integers and let \( s \in \mathbb{Z}_r^+ \) satisfy \( s' = 1 \). We define the graph \( Y(s; t, r) \) to have vertex set \( \{v_i^j : i \in \mathbb{Z}_t, j \in \mathbb{Z}_r \} \) and edges of the form \( v_i^j v_i^{j+1}, v_i^j v_i^{j+s} \), \( (i \in \mathbb{Z}_t, j \in \mathbb{Z}_r) \). (These graphs belong to a more general family of 4-valent graphs admitting a \( \frac{1}{2} \)-transitive group action studied in [13].) It is easily checked that the permutations \( \rho, \sigma \) mapping according to the rules (1) and (2), respectively, and the permutation \( \tau \) mapping according to the rule

\[
v_i^j \tau = v_i^{-j+s'-1}, \quad i \in \mathbb{Z}_t, \quad j \in \mathbb{Z}_r.
\]

are automorphisms of \( Y(s; t, r) \). (Note that \( Y(s; t, r) \) is the Cayley graph of the group \( \langle \rho, \sigma \rangle \cong \mathbb{Z}_r \times \mathbb{Z}_t \) with respect to the set of generators \( \{\sigma, \sigma \rho^s\} \).)

Moreover, the group \( \langle \rho, \sigma, \tau \rangle \) acts \( \frac{1}{2} \)-arc-transitively on \( Y(s; t, r) \).

We are now ready to prove the existence of weakly flag-transitive configurations whose kernels are even length polygons by identifying certain \( \frac{1}{2} \)-arc-transitive graphs in the family of graphs \( Y(s; t, r) \).

**Theorem 3.3** Let \( p \equiv 1 (\text{mod } 3) \) be a prime and let \( s \in \mathbb{Z}_{2p}^+ \) satisfy \( s' \neq 1 \) and \( s^3 = -1 \). Then \( Y(s; 6, 2p) \) is a bipartite \( \frac{1}{2} \)-arc-transitive graph with girth 6 and radius 2p. The corresponding configuration \( D = D(s; 6, 2p) \) is thus a weakly flag-transitive \( (12p) \)-configuration and its kernel is the \( 2p \)-gon.

**Proof.** For each \( i \in \mathbb{Z}_6 \) let \( W_i = \{v_i^j : j \in \mathbb{Z}_{2p} \} \). The graph \( Y = Y(s; 6, 2p) \) is clearly bipartite with the sets \( W_0 \cup W_2 \cup W_4 \) and \( W_1 \cup W_3 \cup W_5 \) being the two parts of bipartition.

Observe first that there are no 4-cycles in \( Y \). (In other words, there is no relation in \( \sigma \) and \( \sigma \rho^s \) of length 4 in the group \( \langle \rho, \sigma \rangle \).) Namely, a 4-cycle in \( Y \) would necessarily have to contain a 2-path of the form \( v_i^j v_i^{j+1} v_i^{j+s} \) for some \( i \in \mathbb{Z}_6 \) and \( j \in \mathbb{Z}_{2p} \). But since \( s' \neq 1 \), we see that \( v_i^j \) and \( v_i^{j+s} \) have no common neighbor other than \( v_i^j \). Since \( Y \cong Cay(\langle \rho, \sigma \rangle, \{\sigma, \sigma \rho^s\}) \) and \( \sigma \) has order 6, it therefore follows that \( Y \) has 6. (For example \( C = v_0^0 v_0^0 v_0^3 v_0^3 v_0^0 \) is a 6-cycle.) In fact, every 6-cycle in \( Y(s; 6, 2p) \) contains precisely one vertex from
each of \( W_i, i \in \mathbb{Z}_6 \). Namely, a 6-cycle of this kind would have to contain three vertices from \( W_i \), and either one vertex from \( W_{i+1} \) and two vertices from \( W_{i-1} \) or one vertex from \( W_{i+1} \) and two vertices from \( W_{i+1} \). For transitivity reasons we may let \( i = 0 \), giving us a 6-cycle of the form \( v_0^0 v_0^0 v_1^0 v_0^1 v_2^0 v_0^0 \) or of the form \( v_0^0 v_0^1 v_1^0 v_0^2 v_2^0 v_0^0 \) for some \( j \in \mathbb{Z}_2 \). But then \( j \in \{2, 2p - 2\} \) in the first case and \( j \in \{2^{-1}, 2p - 2^{-1}\} \) in the second case. Both are impossible as \( 2 \notin \mathbb{Z}_2 \). Consequently, no 2-path with a central vertex in \( W_i \) and either endvertex in \( W_{i+1} \) or both endvertices in \( W_{i+1} \) is contained in a 6-cycle. On the other hand, every 2-path connecting three neighboring sets \( W_i, W_{i+1}, W_{i+2} \) is contained in a 6-cycle. For transitivity reasons it suffices to verify this statement for all such 2-paths with central vertex, say \( v \). For example, the 6-cycles \( v_0^1 v_1^1 v_3^1 v_5^1 v_0^1 v_2^1 \) and \( v_0^1 v_1^1 v_3^1 v_5^1 v_0^1 v_2^1 \) respectively, contain the corresponding four 2-paths \( v_0^1 v_2^1, v_0^1 v_0^1 v_2^1, v_0^1 v_2^1, v_0^1 v_2^1 \) of that kind.

We may now easily deduce that the sets \( W_i, i \in \mathbb{Z}_6 \), form an imprimitivity block system of \( \text{Aut} Y \). Let \( \alpha \in \text{Aut} Y \) and \( i \in \mathbb{Z}_6 \). Assume that \( W_i \cap W_i \alpha \neq \emptyset \) and let \( v \in W_i \cap W_i \alpha \). In view of the facts about 2-paths in \( Y \) discussed above, it follows that \( (W_i \cap N_v^2(\alpha)) \subseteq W_i \). Continuing this way we see that \( (W_i \cap N_v^2(\alpha)) \subseteq W_i \) for each \( k \) and thus \( W_i \alpha = W_i \). Hence \( W_i \) is a block of \( \text{Aut} Y \).

Finally, we prove that \( Y \) is \( 1/2 \)-arc-transitive by showing that \( \text{Aut} Y = \langle \rho, \sigma, \tau \rangle \). Let \( \alpha \) be an arbitrary automorphism of \( Y \) fixing \( v_0^0 \). We claim that \( \alpha \) fixes each of the sets \( W_i \). Assuming the contrary, we must have that \( \alpha \) interchanges \( W_i \) and \( W_{i-1} \) for each \( i \in \mathbb{Z}_6 \). Consequently, \( \alpha \) interchanges the sets \( N(v_0^0) \cap W_i = \{v_0^0, v_1^0\} \) and \( N(v_0^0) \cap W_5 = \{v_3^0, v_5^0\} \) and so \( \alpha \) interchanges the sets \( N(v_0^0) \cap W_5 \setminus \{v_0^0\} = \{v_0^0, v_5^{-1}\} \) and \( N(v_0^0) \cap W_5 \setminus \{v_0^0\} = \{v_0^0, v_5^{-1}\} \). Hence \( \alpha \) interchanges the sets \( N(v_0^0, v_1^0) \cap W_5 = \{v_1^0, v_1^1, v_2^1\} \) and \( N(v_0^0, v_1^0) \cap W_5 = \{v_2^0, v_5^{-1}, v_5^0, v_5^1\} \) and hence the sets \( \{v_1^0, v_1^1\} \) and \( \{v_5^{-1}, v_5^0\} \). Continuing this way we can see that, for each \( j \in \mathbb{Z}_2 \), \( \alpha \) interchanges the sets \( \{v_j^0, v_0^{-j}\} \) and \( \{v_0^j, v_0^{-j}\} \), as well as the sets \( \{v_j^1, v_1^{-j}\} \) and \( \{v_5^j, v_5^{-j}\} \). Consequently, \( \alpha \) interchanges the sets \( N(v_0^j, v_0^{-j}) \cap W_5 = \{v_0^j, v_5^{-j}, v_5^j, v_5^{-j}\} \) and \( N(v_0^j, v_0^{-j}) \cap W_5 = \{v_0^j, v_5^{-j}, v_5^j, v_5^{-j}\} \). But then using the formula for \( \alpha \) on \( W_5 \) we obtain that the former of the two sets must coincide with the set \( \{v_0^j, v_5^{-j}, v_5^j, v_5^{-j}\} \). This forces \( s^{10} = \pm 1 \) and, since \( s^3 = -1 \), we find that \( s = \pm 1 \), a contradiction. We have thus shown that \( \alpha \) fixes each of the sets \( W_i, i \in \mathbb{Z}_6 \) and so it is easily seen that \( \alpha \in \langle \tau \rangle \) and consequently \( \text{Aut} Y = \langle \rho, \sigma, \tau \rangle \). This proves that \( Y \) is
\[ \frac{1}{2} \text{-arc-transitive}. \] We conclude that \( \mathcal{D}(s; 6, 2p) \) is a desired configuration. \( \blacksquare \)

We remark that the smallest graph in the above family of graphs is \( Y(3; 6, 14) \) and so the corresponding \((84)_4\)-configuration \( \mathcal{D}(3; 6, 14) \) is the smallest known weakly flag-transitive configuration whose kernel is an even length polygon, more precisely its kernel is the 14-gon.

We now turn to the existence and construction problem for weakly flag-transitive non-self-polar configurations. In the theorem below we give a complete classification of all those triples \((s, t, r)\) for which \( \mathcal{C}(s, t, r) \) is a non-self polar weakly flag-transitive configuration.

**Theorem 3.4** Let \( t \geq 3 \) be an integer, \( r \geq 3 \) be an odd integer and let \( s \in \mathbb{Z}_r^+ \) satisfy \( s^4 = \pm 1 \) and \( s^2 \neq \pm 1 \). The graph \( X(s, t, r) \) gives rise to a non-self polar weakly flag-transitive \((\frac{r^2}{2})_4\)-configuration \( \mathcal{C}(s, t, r) \) if and only if one of the following conditions holds true.

(i) \( t = 4 \) and \( s^4 = 1, 1 + s + s^2 + s^3 \neq 0 \) and \( 1 - s + s^2 - s^3 \neq 0 \); or

(ii) \( t = 4 \) and \( s^4 = -1 \); or

(iii) \( t = 4k, k \geq 2 \); or

(iv) \( t = 2(2k + 1), k \geq 1, \) and \( s^4 = -1 \).

**Proof.** The conditions (i)-(iv) are clearly necessary. First, \( t \) must be even in order for \( X = X(s; t, r) \) to be bipartite. Next if \( t = 4 \) and \( s^4 = 1 \), then \( X \) has girth 4 if either \( 1 + s + s^2 + s^3 = 0 \) or \( 1 - s + s^2 - s^3 = 0 \), as \( v_0^{0, 1}v_1^{1, 2}v_2^{1, 3}v_3^{1, 4}v_0^{0, 1} \) is a 4-cycle in the first case and \( v_0^{0, 1}v_1^{1, 2}v_2^{1, 3}v_3^{1, 4}v_0^{0, 1} \) is a 4-cycle in the second case. On the other hand, if \( t = 2(2k + 1), k \geq 1, \) and \( s^4 = 1 \), then \( \sigma^{2k+1} \) is an involution interchanging the two bipartition sets.

To prove sufficiency of conditions (i)-(iv), we proceed as follows. Letting \( W_i = \{ v_i^j : j \in \mathbb{Z}_r \} \) for each \( i \in \mathbb{Z}_r \), the two parts of bipartition are \( B_0 = W_0 \cup W_2 \cup \ldots \cup W_{t-2} \) and \( B_1 = W_1 \cup W_3 \cup \ldots \cup W_{t-1} \). Since weak flag-transitivity \( X = X(s; t, r) \) follows directly from Theorem 3.2, only two things need checking (under the assumption that one of conditions (i)-(iv) holds).

First, we have to prove that \( X \) has no 4-cycles, and second, that \( X \) has no involutions interchanging \( B_0 \) and \( B_1 \). Since, by assumption, \( s^2 \neq \pm 1 \) we clearly have no 4-cycles in \( X \) for \( t > 4 \). For \( t = 4 \), the existence of a 4-cycle corresponds to a relation of the form \( 1 \pm s \pm s^2 \pm s^3 = 0 \) in \( \mathbb{Z}_r^+ \). By computation (we omit the tedious details), we may see that this can only
happen when \( s^4 = 1 \) and when, in addition, either \( 1 + s + s^2 + s^3 = 0 \) or \( 1 - s + s^2 - s^3 = 0 \). As for the nonexistence of involutions interchanging \( B_0 \) and \( B_1 \), note that \( \text{Aut} X = \langle \rho, \sigma, \tau \rangle \) by Theorem 3.2. Hence the automorphisms which interchange \( B_0 \) and \( B_1 \) are of the form \( \sigma^i \rho^j \) or \( \sigma^i \rho^j \tau \) for some \( i = 2k + 1 \in \mathbb{Z}_t \) and \( j \in \mathbb{Z}_r \).

By computation, \( \rho^{-1} = \sigma^{-1} \rho \sigma = \rho^*, \) and \( \rho^* = \rho^{-1} \). On the other hand \( \sigma \) and \( \tau \) commute. Thus \( \rho^i \sigma^i = \sigma^i \rho^i \sigma^i \) for all \( i \in \mathbb{Z}_t \) and \( j \in \mathbb{Z}_r \). Now \( (\sigma^i \rho^j)^2 = \sigma^{2i} \rho^{(1+4i)} \) and for this to be identity we would have to have \( \sigma^{2i} = 1 = \rho^{j(1+4i)} \). But this is impossible as \( i \) is odd. A similar contradiction is obtained for automorphisms of the form \( \sigma^i \rho^j \tau \), where \( i = 2k + 1 \in \mathbb{Z}_t \) and \( j \in \mathbb{Z}_r \). This completes the proof of Theorem 3.4. ■

Let us mention that the smallest non-self polar weakly flag-transitive configuration obtained by the above theorem is \( C(2; 4, 17) \) with 34 points, whereas the smallest triangle-free non-self polar weakly flag-transitive configuration is \( C(3; 8, 17) \). In Figure 1 the configuration \( C(2; 4, 17) \) is shown.

### Appendix

Blocks of the four smallest known configurations with prescribed properties are as follows

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Block</th>
<th>Modulo</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(2; 6, 9) )</td>
<td>((1 -1 2' -2'))</td>
<td>((4' -4' 1'' -1''))</td>
</tr>
<tr>
<td>( D(3; 6, 14) )</td>
<td>((0 1 1' 4'))</td>
<td>((1' 6' 0' 1''))</td>
</tr>
<tr>
<td>( C(2; 4, 17) )</td>
<td>((1 3 0' 4'))</td>
<td>((4 5 0' 9'))</td>
</tr>
<tr>
<td>( C(3; 8, 17) )</td>
<td>((1 -1 3' -3'))</td>
<td>((8' -8' 7'' -7''))</td>
</tr>
</tbody>
</table>

where for example, notation \(-4'' \mod 17\) stands for \(17 - 4 + 2 \times 17 = 47\) and \(\(1 -1 2' -2'\) \mod 9\) stands for the nine blocks

\[
\begin{align*}
(1 8 11 16) & \quad (2 0 12 17) & \quad (3 1 13 9) & \quad (4 2 14 10) & \quad (5 3 15 11) \\
(6 4 16 12) & \quad (7 5 17 13) & \quad (8 6 9 14) & \quad (0 7 10 15)
\end{align*}
\]

(compare Figure 1).

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Figure 1: The smallest known weakly flag-transitive non-self polar configuration $\mathcal{C}(2; 4, 17)$
References


