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Abstract

An automorphism $\sigma$ of a finite simple graph $\Gamma$ is an adjacency automorphism if for every vertex $x \in V(\Gamma)$, either $\sigma x = x$ or $\sigma x$ is adjacent to $x$ in $\Gamma$. An adjacency automorphism fixing no vertices is a shift. A connected graph $\Gamma$ is strongly adjacency-transitive (uniquely shift-transitive, respectively) if there is, for every pair of adjacent vertices $x, y \in V(\Gamma)$, an adjacency automorphism (a unique shift, respectively) $\sigma \in \text{Aut} \, \Gamma$ sending $x$ to $y$. The action graph $\Gamma = \text{ActGraph}(G, X, S)$ of a group $G$ acting on a set $X$, relative to an inverse-closed nonempty subset $S \subseteq G$, is defined as follows: the vertex-set of $\Gamma$ is $X$, and two different vertices $x, y \in V(\Gamma)$ are adjacent in $\Gamma$ if and only if $y = sx$ for some $s \in S$.

A characterization of strongly adjacency-transitive graphs in terms of action graphs is given. A necessary and sufficient condition for cartesian products of graphs to be uniquely shift-transitive is proposed, and two questions concerning uniquely shift-transitive graphs are raised.

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1 Strongly adjacency-transitive graphs

The groups and graphs considered in this paper are finite, the graphs are simple and undirected. We refer the reader to [8] for the results on permutation groups.

An automorphism $\sigma \in \text{Aut } \Gamma$ is an adjacency automorphism of a graph $\Gamma$ if for every vertex $x \in V(\Gamma)$, one of the following holds: either $\sigma x = x$ or $\sigma x$ is adjacent to $x$ in $\Gamma$.

The graph $\Gamma$ is adjacency-transitive if there exists, for every pair of vertices $x, y \in V(\Gamma)$, a sequence of adjacency automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_k \in \text{Aut } \Gamma$ such that $\sigma_1 \sigma_2 \cdots \sigma_k x = y$. It is equivalent to say that the adjacency automorphisms of $\Gamma$ generate a transitive group on the vertex-set $V(\Gamma)$.

If, in addition, for every pair of adjacent vertices $x, y \in V(\Gamma)$ there exists an adjacency automorphism $\sigma \in \text{Aut } \Gamma$ sending $x$ to $y$, then $\Gamma$ is strongly adjacency-transitive. The notions of an adjacency automorphism and adjacency-transitivity in graphs were introduced and studied in [9].

In the sequel we denote by $\sim_{\Gamma}$ the vertex adjacency relation in a graph $\Gamma$.

**Proposition 1.1** Every arc-transitive graph having a nontrivial adjacency automorphism is strongly adjacency-transitive.

**Proof.** Let $\Gamma$ be an arc-transitive graph: by definition, for every two ordered pairs of adjacent vertices $(u, v)$ and $(u', v')$ in $\Gamma$, there exists an automorphism $\sigma \in \text{Aut } \Gamma$ such that $\sigma u = u'$ and $\sigma v = v'$. If $\rho \in \text{Aut } \Gamma$ is a nontrivial adjacency automorphism, there exists a vertex $v \in V(\Gamma)$ for which $\rho v \neq v$, thus $v \sim_{\Gamma} \rho v$. For an arbitrary pair of adjacent vertices $x, y \in V(\Gamma)$ denote by $\sigma$ the automorphism of $\Gamma$ sending $v$ to $x$ and $\rho v$ to $y$. Then the conjugate $\sigma \rho \sigma^{-1}$ is an adjacency automorphism of $\Gamma$ sending $x$ to $y$. So $\Gamma$ is strongly adjacency-transitive. □

We characterize strongly adjacency-transitive graphs in terms of action graphs.

The action graph

$$ \Gamma = \text{ActGrph}(G, X, S) $$

of a group $G$ acting on a set $X$, relative to an inverse-closed nonempty subset $S \subseteq G$, is defined as follows (see [1]): the vertex-set of $\Gamma$ is $X$, and two different vertices $x, y \in V(\Gamma)$ are adjacent in $\Gamma$ if and only if $y = sx$ for some $s \in S$. We refer the reader to [4] for the notion of action digraphs and its application, and to [5] for an implementation of the action graph construction.
Theorem 1.2 A graph \( \Gamma \) is strongly adjacency-transitive if and only if it is (isomorphic to) an action graph \( \text{ActGrph}(G, X, S) \), where \( G \) is a group acting transitively and faithfully on the set \( X \) and \( S \) is a subset in \( G \) satisfying the following conditions: \( S = S^{-1} \), \( S \) generates \( G \) and \( S \) is a union of conjugacy classes in \( G \).

Proof. Let \( \Gamma \) be a strongly adjacency-transitive graph. Define \( X = V(\Gamma) \) and let \( S \subseteq \text{Aut} \Gamma \) be the set of adjacency automorphisms of \( \Gamma \). Obviously, \( S = S^{-1} \) holds. The subgroup \( \langle S \rangle \leq \text{Aut} \Gamma \) generated by \( S \) acts transitively and faithfully on \( X \), and two different vertices \( x, y \in X \) are adjacent in \( \Gamma \) if and only if there is an adjacency automorphism \( s \in S \) such that \( y = sx \). So we have

\[
\mathbf{\Gamma} = \text{ActGrph}(\langle S \rangle, X, S),
\]
and \( S \) is a union of conjugacy classes in \( \langle S \rangle \) since the set \( S \) is closed under conjugation in \( \text{Aut} \Gamma \) by [9, Proposition 2.4].

To prove the opposite assertion, let \( G \) be a group acting transitively and faithfully on \( X \) and let \( G \) be generated by a subset \( S \subseteq G \), where \( S \) is a union of conjugacy classes in \( G \) such that \( S = S^{-1} \). Denote by \( \Gamma \) the action graph \( \Gamma = \text{ActGrph}(G, X, S) \). First we prove that \( G \) is a group of automorphisms of \( \Gamma \): for an arbitrary \( g \in G \) we have

\[
x \sim_{\Gamma} y \iff x \neq y \text{ and } y = sx, \; s \in S
\]

\[
\iff gx \neq gy \text{ and } gy = g(sx), \; s \in S
\]

\[
\iff gx \neq gy \text{ and } gy = (gs g^{-1})gx, \; s \in S
\]

\[
\iff gx \neq gy \text{ and } gy = tgx, \; t \in S
\]

\[
\iff gx \sim_{\Gamma} gy.
\]

So \( g \) is an automorphism of \( \Gamma \) and \( G \leq \text{Aut} \Gamma \). Besides, if \( s \in S \) and \( x \in V(\Gamma) \), then either \( sx = x \), or \( sx \neq x \) implying \( x \sim_{\Gamma} sx \). Thus \( S \) is a set of adjacency automorphisms of \( \Gamma \). Since \( S \) generates the vertex-transitive subgroup \( G \) in \( \text{Aut} \Gamma \), the graph \( \Gamma \) is strongly adjacency-transitive.

Defining the action graph (1) to be quasiabelian if \( S \) is a union of conjugacy classes, one may rephrase Theorem 1.2.

Proposition 1.3 A graph is strongly adjacency-transitive if and only if it is (isomorphic to) a connected vertex-transitive quasiabelian action graph.

We give two corollaries of Theorem 1.2. Recall that the Cayley graph \( \Gamma = \text{Cay}(G, S) \) is defined for an arbitrary group \( G \) and a subset \( S \subseteq G \) satisfying
and $S = S^{-1}$: the vertex-set of $\Gamma$ is $G$, and adjacency in $\Gamma$ is given by $g \sim_\Gamma gs$ for all $g \in G$ and all $s \in S$. A quasiabelian Cayley graph is a Cayley graph $\Gamma = \text{Cay}(G, S)$, where $S$ is a union of conjugacy classes in $G$. (See [7] and [9, 10] for results on quasiabelian Cayley graphs, and [3] under the equivalent notion of normal Cayley graphs.)

**Corollary 1.4** [9, Proposition 2.1] Every connected quasiabelian Cayley graph is strongly adjacency-transitive.

**Corollary 1.5** Every connected Cayley graph of an abelian group is strongly adjacency-transitive.

We conclude this section with the following observations.

**Proposition 1.6** There are quasiabelian Cayley graphs that are not Cayley graphs of abelian groups.

**Proof.** Define the quasiabelian Cayley graph $\Gamma = \text{Cay}(S_4, T)$ of the symmetric group $S_4$ relative to the conjugacy class $T$ of all 4-cycles in $S_4$. One can check that $\text{Aut} \Gamma \cong S_4 \not\simeq S_2$, and that no abelian subgroup in $\text{Aut} \Gamma$ is regular. (For instance, compute the order $|\text{Aut} \Gamma| = 1152$ using [5], and proceed with elementary group-theoretic arguments.) Thus $\Gamma$ is not a Cayley graph of an abelian group. ■

**Proposition 1.7** There exist strongly adjacency-transitive Cayley graphs that are not quasiabelian Cayley graphs.

**Proof.** By [9, p. 325], the triangle graph $T_7$ is an adjacency-transitive Cayley graph that is not a quasiabelian Cayley graph. Since its automorphism group has rank 3, the graph $T_7$ is arc-transitive. Proposition 1.1 implies $T_7$ is strongly adjacency-transitive. ■

## 2 Uniquely shift-transitive graphs

A shift of a graph is an adjacency automorphism fixing no vertices (see [3, Definition 3.4]). Shifts are easily found in Cayley graphs of abelian groups, as in the wider class of quasiabelian Cayley graphs: if $\Gamma = \text{Cay}(G, S)$, where $S$ is a union of conjugacy classes in $G$, then the right multiplication by $s \in S$ of elements in $G$ induces a shift of $\Gamma$ (see [9, Proof of Proposition 2.1]), and the same holds for the left multiplication by $s$. 

4
We call a graph \( \Gamma \) shift-transitive if there exists, for every pair of vertices \( x, y \in V(\Gamma) \), a sequence of shifts \( \sigma_1, \sigma_2, \ldots, \sigma_k \in \text{Aut} \Gamma \) such that \( \sigma_1 \sigma_2 \cdots \sigma_k x = y \). If, in addition, for every pair of adjacent vertices \( x, y \in V(\Gamma) \) there exists exactly one (at least one, resp.) shift \( \sigma \in \text{Aut} \Gamma \) sending \( x \) to \( y \), then \( \Gamma \) is uniquely shift-transitive (strongly shift-transitive, resp.).

Observe that the valency of a uniquely shift-transitive graph \( \Gamma \) equals the number of shifts in \( \text{Aut} \Gamma \).

The cycle \( C_4 \) is not uniquely shift-transitive. Written as \( P_2 \times P_2 \), it presents the fundamental obstruction for the cartesian product to preserve uniquely shift-transitivity, as stated in the following theorem. (We refer the reader to [6] for the theorem on unique prime cartesian factorization of connected graphs.)

**Theorem 2.1** A graph \( \Gamma \) is uniquely shift-transitive if and only if in the prime cartesian factorization of \( \Gamma \), all factors are uniquely shift-transitive and at most one factor is isomorphic with \( P_2 \).

To prove Theorem 2.1 we need some auxiliary results. We omit the justifications of the first two.

**Lemma 2.2** If \( \gamma \in \text{Aut} \Gamma \) and \( \delta \in \text{Aut} \Delta \) are shifts of \( \Gamma \) and \( \Delta \), respectively, then the automorphisms \( \gamma \times \text{id} \Delta \) and \( \text{id} \Gamma \times \delta \) are shifts of the cartesian product \( \Gamma \times \Delta \).

The automorphisms \( \gamma \times \text{id} \Delta \) and \( \text{id} \Gamma \times \delta \) in Lemma 2.2 are called cartesian shifts along the factors \( \Gamma \) and \( \Delta \), respectively.

**Proposition 2.3** Let \( \Gamma \) and \( \Delta \) be shift-transitive graphs. Then

(a) the cartesian product \( \Gamma \times \Delta \) is shift-transitive;

(b) if \( \Gamma \) or \( \Delta \) is not uniquely shift-transitive then neither is \( \Gamma \times \Delta \).

**Lemma 2.4** Let \( \Gamma \) and \( \Delta \) be connected graphs and let \( \sigma \in \text{Aut} (\Gamma \times \Delta) \) be a shift. Then

(a) if \( \sigma \) fixes setwise one of the fibers \( \{c\} \times V(\Delta) \), \( c \in V(\Gamma) \), then \( \sigma \) fixes setwise each of them, and \( \sigma = \text{id} \Gamma \times \delta \) for some shift \( \delta \in \text{Aut} \Delta \);

(b) if \( \sigma(\{c\} \times V(\Delta)) \cap (\{c\} \times V(\Delta)) = \emptyset \) for some \( c \in V(\Gamma) \), then \( \sigma \) fixes setwise each of the fibers \( V(\Gamma) \times \{d\}, d \in V(\Delta) \), and \( \sigma = \gamma \times \text{id} \Delta \) for some shift \( \gamma \in \text{Aut} \Gamma \);
(c) if there is a vertex \( u \) in a fiber \( \{ c \} \times V(\Delta) \), \( c \in V(\Gamma) \), such that 
\( u, \sigma u, \sigma^2 u \in \{ c \} \times V(\Delta) \), then \( \sigma = \id_{\Gamma} \times \delta \) for some shift \( \delta \in \Aut \Delta \).

**Proof.** Denote by \( \rho \) the canonical projection of \( V(\Gamma) \times V(\Delta) \) onto \( V(\Delta) \).

(a) Let the shift \( \sigma \) fix setwise the fiber \( F = \{ c \} \times V(\Delta) \) for some \( c \in V(\Gamma) \), and let \( c' \in V(\Gamma) \) be adjacent to \( c \) in \( \Gamma \).

Choose an arbitrary vertex \( d \in V(\Delta) \). Then \( \sigma(c,d) = (c,d') \) for some neighbour \( d' \in V(\Delta) \) of \( d \). Let \( \sigma(c',d) = (c'',d') \), where \( c'' \in V(\Gamma) \) and \( d' \in V(\Delta) \). Suppose \( \sigma(c',d) \notin F' = \{ c' \} \times V(\Delta) \). Then, since \( \sigma \) is a shift of \( \Gamma \times \Delta \), we have \( d'' = d \) and the vertex \( c'' \) is adjacent to \( c' \) in \( \Gamma \). The vertices \( (c,d) \) and \( (c',d) \) are adjacent in \( \Gamma \times \Delta \), and so are their images \( \sigma(c,d) = (c,d') \) and \( \sigma(c',d) = (c'',d) \). Thus \( c = c'' \), giving \( \sigma(c',d) \in F \), a contradiction to \( \sigma F = F \). So

\[
\sigma(c',d) \in F',
\]

whence \( d' = c' \) and \( d'' \) is adjacent to \( d \) in \( \Delta \). The same adjacency argument as above gives \( d'' = d' \), i.e.

\[
\rho \sigma(c,d) = \rho \sigma(c',d).
\]

One infer from (2) that the shift \( \sigma \) fixes setwise the fiber \( F' \). The connectedness of \( \Gamma \) implies the shift \( \sigma \) fixes setwise all fibers over \( \Gamma \). Adding (3), one gets \( \sigma = \id_{\Gamma} \times \delta \) for some shift \( \delta \in \Aut \Delta \).

(b) Let \( F = \{ c \} \times V(\Delta) \) and suppose \( \sigma F \cap F = \emptyset \). Verify that the shift \( \sigma \) moves every fiber \( F' = \{ c' \} \times V(\Delta) \) over \( \Gamma \) into a fiber over \( \Gamma \), and that \( \sigma F' \cap F' = \emptyset \).

(c) Denote by \( F \) the fiber \( V(\Gamma) \times \{ \rho \sigma u \} \). Check that the intersection \( \sigma(F) \cap F \) is empty, then apply (b).

\[
\text{Proposition 2.5 Let } \Gamma \text{ and } \Delta \text{ be connected graphs and let } \sigma \in \Aut (\Gamma \times \Delta) \text{ be a shift which is not cartesian along } \Gamma \text{ or } \Delta. \text{ Then each of the graphs } \Gamma \text{ and } \Delta \text{ is isomorphic to a cartesian product with a } P_2 \text{ factor.}

\]

**Proof.** Suppose the shift \( \sigma \in \Aut (\Gamma \times \Delta) \) is not cartesian along \( \Gamma \) or \( \Delta \). Then \( \Gamma \) and \( \Delta \) have order at least 2. Fix an arbitrary vertex \( c \in V(\Gamma) \), then define the \( c \)-fiber

\[
F = \{ c \} \times V(\Delta)
\]
and its subset
\[ A = \{ u \in F \mid \sigma u \in F \}. \]

By Lemma 2.4(a,b) we have \( A \neq \emptyset \) and \( A \neq F \). We will show that \( \Delta \) is isomorphic to the cartesian product \( \Omega \times P_2 \), where \( \Omega \) is the subgraph in \( \Gamma \times \Delta \) induced by \( A \). Define
\[ B = F \setminus A = \{ u \in F \mid \sigma u \notin F \}. \]

Then \( \sigma A \subseteq B \) by Lemma 2.4(c). Moreover, there are no edges between \( A \) and \( B \) except the matching amongst \( A \) and \( \sigma A \) induced by the shift \( \sigma \): if \( v \in A \) is adjacent to \( w \in B \) and \( w \neq \sigma v \), then the vertices \( \sigma v \in F \setminus \{ w \} \) and \( \sigma w \notin F \) are not adjacent in \( \Gamma \times \Delta \), a contradiction.

We now prove \( B = \sigma A \). Suppose \( B \neq \sigma A \) and fix a vertex \( w \in B \setminus \sigma A \) adjacent to a vertex \( v \in \sigma A \). Then \( \sigma^{-1}w \notin F \), so the image \( \sigma^{-1}w \) is not adjacent to \( \sigma^{-1}v \in A \), a contradiction. Therefore \( B = \sigma A \), and \( \Delta \simeq \Omega \times P_2 \).

One shows similarly that \( \Gamma \simeq \Pi \times P_2 \) for some (connected) graph \( \Pi \). □

**Remark.** It follows from the proof of Proposition 2.5 that
\[ \Gamma \times \Delta \simeq (\Omega \times \Pi) \times C_4. \]

The shift \( \sigma \) is cartesian along the factors of the right factorization in (4),
\[ \sigma = \text{id}_{\Omega \times \Pi} \times \phi, \]
where \( \phi \in \text{Aut} C_4 \) is a shift of order 4.

**Proposition 2.6** The cartesian product \( \Gamma \times \Delta \) of two graphs \( \Gamma \) and \( \Delta \) is shift-transitive if and only if \( \Gamma \) and \( \Delta \) are shift-transitive.

**Proof.** Proposition 2.3(a) settles the "if" implication. We prove the "only if" part. Let \( \Gamma \times \Delta \) be shift-transitive. Factorize \( \Delta \simeq (P_2)^s \times \Delta' \), where \( s \geq 0 \) and \( \Delta' \) has no \( P_2 \) factor in its cartesian factorization. Then
\[ \Gamma \times \Delta \simeq (\Gamma \times (P_2)^s) \times \Delta' = \Sigma. \]

Put \( \Gamma' = \Gamma \times (P_2)^s \). If \( \Delta' \) is not shift-transitive, then the shift-transitive graph \( \Sigma \) has a shift which is not cartesian along the factors \( \Gamma' \) or \( \Delta' \). By Proposition 2.5, \( \Delta' \) is a cartesian product with a \( P_2 \) factor, a contradiction. Thus \( \Delta' \) is shift-transitive, and so is \( \Delta \) by Proposition 2.3(a). □

Theorem 2.1 is a corollary of Propositions 2.5 and 2.6. It leads to abundance of uniquely shift-transitive graphs among Cayley graphs of abelian groups.
Corollary 2.7 The cartesian product of cycles $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ is uniquely shift-transitive if and only if there are no 4-cycles involved.

Corollary 2.8 Every abelian group of order not divisible by 4 admits a uniquely shift-transitive Cayley graph.

3 Two questions

Examples of uniquely shift-transitive graphs can be found among Cayley graphs of abelian groups: besides the cartesian product of cycles as in Corollary 2.7 we have, for instance, the Möbius ladder $M_n$, $n \geq 4$ (see also [9, Section 3]).

Proposition 3.1 Let $\Gamma = \text{Cay}(G, S)$ be a quasiabelian Cayley graph of a nonabelian group $G$, where the generating set $S$ is an inverse-closed union of conjugacy classes and $1 \notin S$. Then $\Gamma$ is not uniquely shift-transitive.

Proof. For $s \in S \setminus Z(G)$, the left and the right multiplication by $s$ induce two different shifts of $\Gamma$ sending the vertex 1 to $s$. 

Thus a uniquely shift-transitive quasiabelian Cayley graph must be a Cayley graph of an abelian group.

Question 1 Is every uniquely shift-transitive Cayley graph a Cayley graph of an abelian group?

If the answer to Question 1 is positive, then every shift $\sigma$ of a uniquely shift-transitive Cayley graph $\Gamma$ arises from the multiplication by a fixed element of an abelian group. Hence $\sigma$ must be semiregular, i.e. all cycles in its cyclic decomposition have same length. This fact may prove useful in approaching the problem.

Question 2 Does there exist a uniquely shift-transitive non-Cayley graph?

The answer to Question 2 is negative in case we omit the uniqueness of the shift acting along an arbitrary edge, according to the following result.

Proposition 3.2 There exist strongly shift-transitive non-Cayley graphs.
Proof. Let $\Gamma = (K(n, k))^c$ be the complement of Kneser’s graph $K(n, k)$, where $n$ and $k$ are two positive integers such that $n = 2k + 1$ and $k \geq 3$. The vertices of $\Gamma$ are the $k$-elements subsets in $I_n = \{1, 2, \ldots, n\}$, and two such $k$-subsets are adjacent in $\Gamma$ if and only if they have nontrivial intersection. By [2], every automorphism of the graph $\Gamma$ arises from a permutation in $S_n$ acting naturally on the $k$-subsets of $I_n$. One can check that the automorphism of $\Gamma$ induced by any $s$-cycle in $S_n$, $k + 2 \leq s \leq 2k - 1$, is a shift of $\Gamma$, and any vertex of $\Gamma$ may be moved to a neighbour by at least two shifts of this kind. However, $\Gamma$ is not a Cayley graph by [2].

References


